CONTRACTION THEORY AS METHOD FOR THE ANALYSIS AND DESIGN OF THE STABILITY OF COLLECTIVE BEHAVIOR IN CROWDS

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ABSTRACT

The modeling of the collective behavior of many characters is an important problem in crowd animation. Such behaviors can be described by solutions of large-scale nonlinear dynamical systems, which are built from many interacting components. The design of the stability properties of such complex multi-component systems has been rarely studied in computer animation. We present an approach for the solution of this problem that is based on Contraction Theory, a framework for the analysis of the stability of complex nonlinear dynamical systems. Based on learning-based realtime capable architecture for the animation of crowds, we demonstrate the application of this novel approach for stability design. We derive conditions guaranteeing the global asymptotic stability of the formation of coordinated navigation behavior of crowds. In addition, we demonstrate that the same approach permits to derive bounds that guarantee the minimal convergence rates of the formation of order in navigating crowds.

KEYWORDS

computer animation, coordination, crowd animation, stability

1. INTRODUCTION

Dynamical systems are frequently applied in crowd animation for the simulation of autonomous and collective behavior of many characters (Musse & Thalmann 2001, Treuille et al. 2006). Some of this work has been inspired by observations in biology, showing that coordinated behavior of large groups of agents, such as flocks of birds, can be modeled as emergent behavior that arises from the dynamical coupling between interacting agents, without requiring an external central mechanism that ensures coordination (Cucker & Smale 2007, Couzin 2009, Camazine et al. 2001). Such models can be analyzed by application of methods from nonlinear dynamics (Pikovsky et al. 2003). The simulation of collective behavior by self-organization in systems of dynamically coupled agents is interesting because it might reduce the computational costs of traditional computer animation techniques, such as scripting or path planning (Treuille et al. 2006). In addition, the generation of collective behavior by self-organization allows to implement spontaneous adaptation to external perturbations or changes in the system architecture, such as the variation of the number of characters (Cucker & Smale 2007). However, the mathematical analysis of the underlying dynamical systems is typically quite complicated. The dynamics describing individual agents is typically highly nonlinear, making a systematic treatment of stability properties often infeasible even for individual characters. In addition, crowd animation requires the dynamic interaction of many such agents. Consequently, the convergence and stability properties of such dynamical systems have been rarely addressed in the context of computer animation. Yet, approaches for a systematic analysis and design of the dynamical properties of crowd animation systems seem highly desirable, since they might permit to guarantee desired system properties and the robustness of the generated behavior under variations of system inputs and the system parameters.

In this paper we introduce Contraction Theory (Lohmiller & Slotine 1998) as a novel framework for the analysis and design of the convergence properties of navigating avatars during self-organized order formation. This framework is applied to simple learning-based animation architecture for the real-time synthesis of interacting characters. The paper is structured as follows: The structure of the animation system is briefly sketched in **section 2**. The dynamics underlying navigation control is described in **section 3**. Subsequently, in **section 4** we introduce some basic ideas of Contraction Theory. The major results of our stability analysis and some demos of their application to crowd animation are described in **section 5**, followed by some conclusions.

2. SYSTEM ARCHITECTURE

Our investigation of the collective dynamics of crowds was based on a learning-based animation system (Giese et al. 2009), see **Fig.1**. Based on motion capture data, we learned spatio-temporal components of sets of different gait types, applying an algorithm for translation-invariant blind source separation (Omlor & Giese 2006). The obtained source components were generated online by nonlinear dynamical systems, whose state dynamics was given by a limit cycle oscillators. The mappings $\sigma_j(t)$ between the stable solution of the nonlinear oscillators and the required source functions were learned by application of kernel methods. Each character is modeled by a single Andronov-Hopf oscillator (Giese et al. 2009), whose solution is mapped nonlinearly onto three source signals. These signals were then superimposed with different linear weights w_{ij} and phase delays τ_{ij} in order to generate the joint angle trajectories $\xi_i(t)$. By blending of the mixing weights and the phase delays, intermediate gait styles were generated. This allowed us to simulate specifically walking along paths with different curvatures, and changes in step length. Interactive behavior of multiple characters can be modeled by making the states of the oscillators and the mixing weights dependent on the behavior of the other characters. Such coupling results in a highly nonlinear system dynamics.

For the crowd animation scenarios in this paper, the heading directions of the characters were controlled by a navigation dynamics that steers the avatars towards goal points, which were placed along parallel straight lines in front of the characters. The heading dynamics was given by a nonlinear first-order differential equation (see (Giese et al. 2009) for details). This control of heading direction was active only during the initial stage of the organization of the crowd, which results in an alignment of the characters along the parallel straight lines, independent of their initial positions and gait phases. For the mathematical stability analysis presented in the following, we neglected the influence of the dynamics of the control of heading direction, focusing on the second phase of order formation when the characters' heading directions are already aligned. In this case, the positions of the characters can be described by a single position variable $z(t)$, see **Fig. 2**.

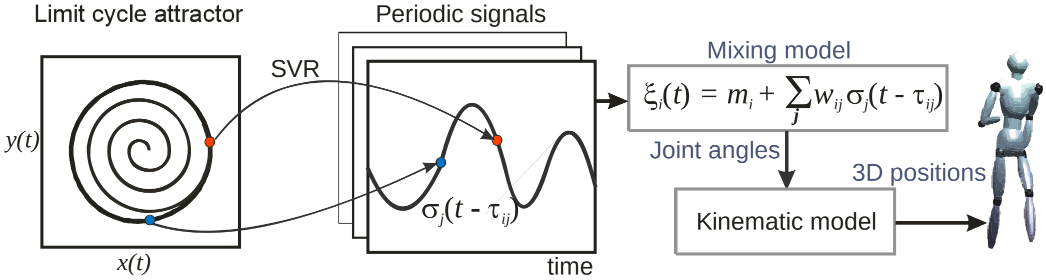

Figure 1. Architecture of the animation system.

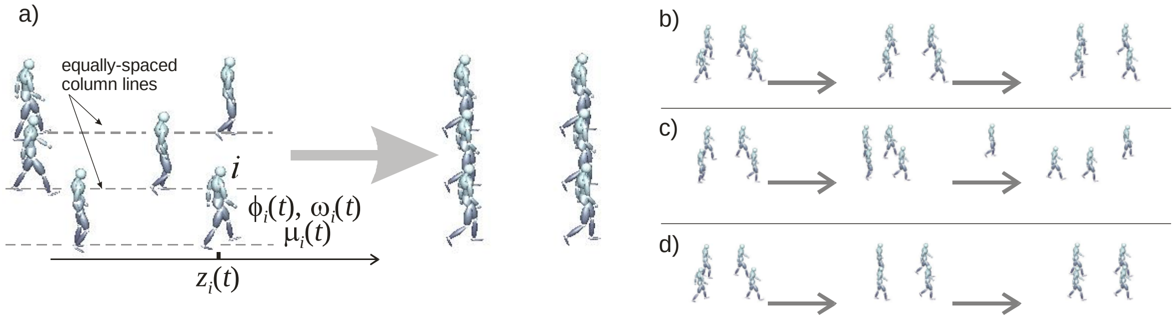

Figure 2. a. Crowd coordination setup: every avatar *i* is characterized by its position $z_i(t)$, the phase $\phi_i(t)$, and the instantaneous eigenfrequency $\omega_i(t) = \varphi_i(t)$ of Andronov-Hopf oscillator, and the step-size scaling parameter $\mu_i(t)$. Self-organizing scenarios: **b.** control of step frequency by distances (convergent case); **c.** same scenario (divergent behavior); **d.** control of step length together with phase synchronization (convergent case).

3. CONTROL DYNAMICS

Beyond the control of heading direction, the analyzed scenarios of order formation in crowds require the control of the following variables: 1) phase of the step cycle and 2) step length. Neglecting the influence of heading direction, where each individual character was modeled by an Andronov-Hopf oscillator with constant equilibrium amplitude ($r_i^* = 1$). For an appropriate choice of parameters, these nonlinear oscillators have a limit cycle that corresponds to a circular trajectory in phase space (Andronov et al. 1987).

In polar coordinates and with instantaneous eigenfrequency ω this dynamics is given by: $\dot{r}(t) = r(t)(1 - r^2(t))$, $\dot{\phi}(t) = \omega$. Control affects the instantaneous eigenfrequency ω of the Andronov-Hopf oscillators and its phase ϕ, while the first equation guarantees that the state stays on the limit cycle ($r(t) = 1, \ \forall t$).

The position z_i of any character along the parallel paths (see **Fig. 2**) fulfils the differential equation $\dot{z}_i(t) = \dot{\phi}_i g(\phi_i)$, where the positive function g determines the propagation speed of the character depending on the phase in the gait cycle. This nonlinear function was determined empirically from the kinematics of the character. By integration of this propagation dynamics one obtains: $z_i(t) = G(\phi_i) + \tilde{c}_i$, with some constant $\tilde{c}_i$ that depends on the initial gait phase of character *i* and with $G(\phi_i) = \int_0^{\phi_i} g(\phi) d\phi$, assuming $G(0) = 0$. Three control rules were implemented:

3.1 Control of step frequency

A simple form of speed control is based on making the frequency of the oscillators $\dot{\phi}_i$ dependent on the behavior of the other characters. Let ω_0 be the equilibrium frequency of the oscillators without interaction, then a simple controller is defined by the differential equation:

$$(1) \quad \dot{\phi}_i(t) = \omega_0 - m_d \sum_{j=1}^{N} K_{ij} [(G(\phi_i(t) + \tilde{c}_i) - (G(\phi_j(t) + \tilde{c}_j) - d_{ij}]$$

The constants d_{ij} define the stable pairwise relative distances in formed order for each pair (*i*, *j*) of characters. The elements of the link adjacency matrix **K** are $K_{ij} = 1$ if characters *i* and *j* are coupled and zero otherwise. In addition, we assume $K_{ii} = 0$. The constant $m_d > 0$ defines the coupling strength.

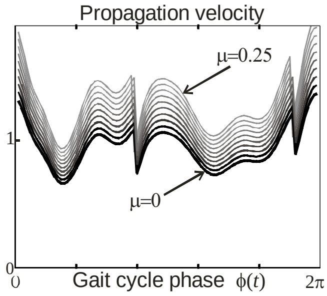

Figure 3. Propagation velocity for 10 different values of step length morphing parameter μ = [0 . . . 0.25] dependent on gait cycle phase $\phi(t)$ and $\omega(t) = 1$. The vertical axis is scaled in order to make the average velocity equal to one for $\mu = 0$ (lowest thick line). This empirical estimates are well approximated by $(1+\mu)g(\phi_i(t))$.

With the Laplacian $\mathbf{L}^d$ of the coupling graph, which is defined by $L_{ij}^d = -K_{ij}$ for $i \neq j$ and $L_{ii}^d = \sum_{j=1}^N K_{ij}$, and constants $c_i = -\sum_{j=1}^N K_{ij} d_{ij}$ and $\hat{c}_i = \sum_{j=1}^N K_{ij}(\tilde{c}_i - \tilde{c}_j)$ the last equation system can be written in vector form:

(2) $\dot{\boldsymbol{\varphi}} = \omega_0 1 - m_d(\mathbf{L}^d G(\boldsymbol{\varphi}) + \mathbf{c} + \hat{\mathbf{c}})$

3.2 Control of step length

Step length was varied by morphing between gaits with short and long steps. A detailed analysis shows that the influence of step length on the propagation could be well captured by simple linear rescaling. If the propagation velocity of avatar *i* is $v_i(t) = \dot{z}_i(t) = \dot{\phi}_i(t)g(\phi_i(t)) = \omega_i(t)g(\phi_i(t))$ for the normal step size, then the velocity for modified step size was well approximated by $v_i(t) = \dot{z}_i(t) = (1+\mu_i)\omega_i(t)g(\phi_i(t))$ with the morphing parameter μ_i. The range of morphing parameters was restricted to the interval $-0.5 < \mu_i < 0.5$, where this linear scaling law was fulfilled with high accuracy. The empirically estimated propagation velocity in heading direction, dependent on gait phase, is shown in **Fig. 3** for different values of the step length morphing parameter μ. Using the same notations as in equation (**1**), this motivates the definition of the following dynamics that models the influence of step length control on the propagation speed:

(3) $\dot{\mathbf{z}} = \dot{\boldsymbol{\varphi}} g(\boldsymbol{\varphi})(1 - m_z(\mathbf{L}^z \mathbf{z} + \mathbf{c}))$

In this equation $\mathbf{L}^z$ signifies the Laplacian of the relevant coupling graph, and m_z the strength of the coupling. For uncoupled characters ($m_z = 0$) this equation is consistent with the definition of propagation speed that was given before.

3.3 Control of step phase:

By defining separate controls for step length and step frequency it becomes possible to dissociate the control of position and step phase of the characters. Specifically, it is interesting to introduce a controller that results in phase synchronization between different characters. This can be achieved by addition of a simple linear coupling term to equation (**1**)

(4) $\dot{\boldsymbol{\varphi}} = \omega_0 1 - m_d(\mathbf{L}^d G(\boldsymbol{\varphi}) + \mathbf{c} + \hat{\mathbf{c}}) - k\mathbf{L}^\phi \boldsymbol{\varphi}$

with $k > 0$ and the Laplacian $\mathbf{L}^\phi$. (All sums or differences of angular variables were computed by modulo 2π).

The mathematical results derived in the following section apply to subsystems of this the complete system dynamics that is given by equations (**3**) and (**4**). In addition, animations are presented for the full system dynamics.

4. CONTRACTION THEORY

Dynamical systems describing the behavior of autonomous characters are essentially nonlinear. In contrast to the linear dynamical systems, a major difficulty of the analysis of stability properties of the nonlinear is that the stability properties of parts usually do not transfer to composite systems. Contraction Theory (Lohmiller & Slotine 1998) provides a general method for the analysis of essentially nonlinear systems, which permits such a transfer, making it suitable for the analysis of complex systems with many components. Contraction Theory characterizes the system stability by the behavior of the differences between solutions with different initial conditions. If these differences vanish exponentially over time, and its solution converges towards a single trajectory, independent from the initial states, the system is called *globally asymptotically stable*. For a general dynamical system of the form

(5) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$

assume that $\mathbf{x}(t)$ is one solution of the system and that $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) + \delta\mathbf{x}(t)$ is a neighboring one with a different initial condition. The function $\delta\mathbf{x}(t)$ is also called virtual displacement. With the Jacobian of the system $\mathbf{J}(\mathbf{x}, t) = \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}}$ it can be shown (Lohmiller & Slotine 1998) that any nonzero virtual displacement decays *exponentially* to zero over time if the symmetric part of the Jacobian $\mathbf{J}_s = (\mathbf{J} + \mathbf{J}^T)/2$ is uniformly negative definite, denoted as $\mathbf{J}_s < 0$, i.e. has negative eigenvalues for all relevant state vectors $\mathbf{x}$. In this case, it can be shown that the norm of the virtual displacement decays at least exponentially to zero, for $t \to \infty$. If the virtual displacement is small enough, then $\delta\dot{\mathbf{x}}(t) = \mathbf{J}(\mathbf{x}, t)\delta\mathbf{x}(t)$, implying through $\frac{d}{dt}\|\delta\mathbf{x}(t)\|^2 = 2\delta\mathbf{x}^T(t)\mathbf{J}_s(\mathbf{x}, t)\delta\mathbf{x}(t)$ the inequality: $\|\delta\mathbf{x}(t)\| = \|\delta\mathbf{x}(0)\| \exp\left[\int_0^t \lambda_{\max}(\mathbf{J}_s(\mathbf{x}, s))ds\right]$.

This decay occurs with a *convergence rate* (inverse timescale) that is bounded from below by the quantity $\rho_c = -\sup_{\mathbf{x},t} \lambda_{\max}(\mathbf{J}_s(\mathbf{x}, t))$, where $\lambda_{\max}(.)$ signifies the largest eigenvalue. This has the consequence that all trajectories converge to a single solution exponentially in time (Lohmiller & Slotine 1998).

An important extension of contraction analysis is applying it to the hierarchically coupled systems (Lohmiller & Slotine 1998). Consider the composite dynamical system:

(6) $\frac{d}{dt}\begin{pmatrix}\mathbf{x}_1 \\ \mathbf{x}_2\end{pmatrix} = \begin{pmatrix}\mathbf{f}_1(\mathbf{x}_1) \\ \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2)\end{pmatrix}$

where the dynamics of the first subsystem is not influenced by the dynamics of the second. Such system is called *hierarchically coupled* and its Jacobian is:

(7) $\mathbf{F} = \begin{pmatrix}\partial\mathbf{f}_1(\mathbf{x}_1)/\partial\mathbf{x}_1 & \mathbf{0} \\ \partial\mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2)/\partial\mathbf{x}_1 & \partial\mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2)/\partial\mathbf{x}_2\end{pmatrix} = \begin{pmatrix}\mathbf{F}_{11} & \mathbf{0} \\ \mathbf{F}_{21} & \mathbf{F}_{22}\end{pmatrix}$

Consider then the smooth dynamics of virtual displacements:

(8) $\frac{d}{dt}\begin{pmatrix}\delta\mathbf{x}_1 \\ \delta\mathbf{x}_2\end{pmatrix} = \begin{pmatrix}\mathbf{F}_{11} & \mathbf{0} \\ \mathbf{F}_{21} & \mathbf{F}_{22}\end{pmatrix}\begin{pmatrix}\delta\mathbf{x}_1 \\ \delta\mathbf{x}_2\end{pmatrix}$

where $\mathbf{F}_{21}$ is bounded. The first subsystem does not depend on the second one, so that $\delta\mathbf{x}_1$ exponentially converges to 0 if $(\mathbf{F}_{11})_s < 0$. Then, $\mathbf{F}_{21}\delta\mathbf{x}_1$ is an exponentially decaying disturbance for the second subsystem. In this case, uniformly negative definite $\mathbf{F}_{22}$ implies exponential convergence of $\delta\mathbf{x}_2$ to an exponentially decaying ball, see (Lohmiller & Slotine 1998) for details of proof. The whole system is then globally exponentially convergent to a single trajectory.

Many systems are not contracting with respect to all dimensions of the state space, but show convergence with respect to a subset of dimensions. Such behavior can be mathematically characterized by *partial contraction* (Wang & Slotine 2005, Park et al. 2009). The underlying idea is to construct an auxiliary system that is contracting with respect to a subset of dimensions (or submanifold) in state space. The major result is the following:

Theorem 1 *Consider a nonlinear system of the form*

(9) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t)$

and assume that the auxiliary system

(10) $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, t)$

is contracting with respect to $\mathbf{y}$ *uniformly for all relevant* $\mathbf{x}$*. If a particular solution of the auxiliary system verifies a specific smooth property, then all trajectories of the original system (**9**) verify this property with exponential convergence. The original system is then said to be partially contracting. (Wang & Slotine 2005).*

A 'smooth property' is a property of the solution that depends smoothly on space and time, such as convergence against a particular solution or a properly defined distance to subspace in the phase space. The proof of the theorem is immediate noticing that the observer-like system (**10**) has $\mathbf{y}(t) = \mathbf{x}(t)$ for all $t \geq 0$ as a particular solution. Since all trajectories of the **y**-system converge exponentially to a single trajectory, this implies that also the trajectory $\mathbf{x}(t)$ verifies this specific property with exponential convergence.

It thus is sufficient to show that the auxiliary system is contracting to prove convergence to a subspace. Let us also assume that the system has a flow-invariant *linear subspace* $\mathcal{M}$, which is defined by the property that trajectories starting in this space always remain in it for arbitrary times ($\forall t$: $\mathbf{f}(\mathcal{M},t) \subset \mathcal{M}$). If matrix $\mathbf{V}$ is an orthonormal projection onto $\mathcal{M}^{\perp}$, then the sufficient condition for global exponential convergence to $\mathcal{M}$ is: $\mathbf{V}\mathbf{J}_s\mathbf{V}^T < 0$, where smaller sign indicates that the matrix is negative definite (Pham & Slotine 2007, Park et al. 2009).

5. RESULTS: STABILITY ANALYSIS FOR DIFFERENT NAVIGATION SCENARIOS

We derived contraction bounds for three scenarios that correspond to the control dynamics with increasing levels of complexity.

5.1 Control of step phase without position control:

The simplest case is the control of the phase of the walkers' step cycles without simultaneous control of the position of the characters. Such simple control already permits to simulate interesting behaviors, such as soldiers synchronizing their step phases (Park et al. 2009). The underlying dynamics is given by (**4**) with $m_d = 0$. For *N* identical dynamical systems with symmetric identical coupling gains *k* this dynamics can be written as.

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + k \sum_{j \in \mathcal{N}_i} (\mathbf{x}_j - \mathbf{x}_i), \quad \forall i = 1, \ldots, N$$

where $\mathcal{N}_i$ defines the index set specifying the neighborhood in the coupling graph, i.e. the other subsystems or characters that are coupled with character *i* (see **Fig.4** for examples).

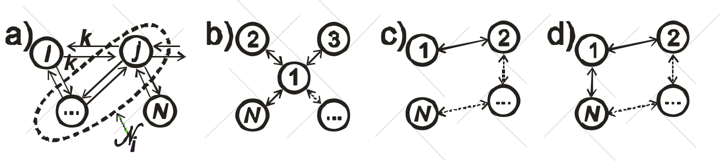

Figure 4. a. Symmetric coupling with coupling constant *k*, $\mathcal{N}_i$ specifying the set of neighbors of *i*; **b.-d.** star, chain, ring coupling schemes.

This type of symmetric coupling, where the interaction forces between subsystems depending only on the differences of the phase variables is called *diffusive coupling*. In this case, the *Laplacian matrix* of the coupling scheme is given by $\mathbf{L} = \mathbf{L}_G \otimes \mathbf{I}_p$, where *p* is the dimensionality of the individual sub-systems, and where $\otimes$ signifies *Kronecker product*. The Laplacian of the coupling graph is the matrix $\mathbf{L}_G$. The system can be then rewritten compactly as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) - k\mathbf{L}\mathbf{x}$ with the concatenated phase variable $\mathbf{x} = [\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T]^T$. The Jacobian of this system is $\mathbf{J}(\mathbf{x},t) = \mathbf{D}(\mathbf{x},t) - k\mathbf{L}$, where the block-diagonal matrix $\mathbf{D}(\mathbf{x},t)$ has the Jacobians of the uncoupled components $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_i, t)$ as entries.

The dynamics has a flow-invariant linear subspace $\mathcal{M}$ that contains the particular solution $\mathbf{x}_1^* = \ldots = \mathbf{x}_N^*$. For this solution all state variables $\mathbf{x}_i$ are identical and thus in synchrony. In addition, for this solution the coupling term in equation (**5**) vanishes, so that the form of the solution is identical with the solution of the uncoupled systems $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i)$. If $\mathbf{V}$ is a projection matrix onto the subspace $\mathcal{M}^{\perp}$, then, the sufficient condition for convergence toward $\mathcal{M}$ is $\mathbf{V}(\mathbf{D}(\mathbf{x},t) - k\mathbf{L})_s \mathbf{V}^T < 0$ (Pham & Slotine 2007, Park et al. 2009). The smaller sign indicating that the matrix is negative definite. With $\lambda_{\mathbf{L}}^+$ signifying the smallest non-zero eigenvalue of symmetric part of the Laplacian $\mathbf{L}_s$ this implies: $\lambda_{\min}(\mathbf{V}(k\mathbf{L})_s \mathbf{V}^T) = k\lambda_{\mathbf{L}}^+ > \sup_{\mathbf{x},t} \lambda_{\max}(\mathbf{D}_s)$

The maximal eigenvalue for the individual dynamical system is $\sup_{\mathbf{x},t} \lambda_{\max}(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x},t))$. The sufficient conditions for global stability of the overall system is given by $k > \sup_{\mathbf{x},t} \lambda_{\max}(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x},t)) / \lambda_{\mathbf{L}}^+$. This implies the minimum convergence rate: $\rho_c = -\sup_{\mathbf{x},t} \lambda_{\max}(\mathbf{V}(\mathbf{D}(\mathbf{x},t) - k\mathbf{L})_s \mathbf{V}^T)$.

For the special case of (**4**) with $m_d = 0$ this implies the sufficient contraction conditions $k > 0$ and $(\mathbf{L}^{\phi})_s \geq 0$. Different topologies of the coupling graphs result in different stability conditions, since for example $\lambda_{\mathbf{L}}^+ = 2(1 - \cos(2\pi/N))$ for symmetric ring coupling, while $\lambda_{\mathbf{L}}^+ = N$ for all-to-all coupling. Where N is the number of characters. See (Wang & Slotine 2005, Park et al. 2009) for details.

In order to validate these theoretical bounds we computed empirical convergence rates ρ^{exper} from our animation system. They were obtained by analyzing the time courses of the virtual displacements $\|\delta x\| \sim \exp(-\rho^{exper} t)$ and approximating them by exponential convergence. The norm of the virtual displacement $\|\delta x\|$ was approximated by the angular dispersion $\hat{R} = (1 - \frac{1}{N}\left|\sum_j e^{i\phi_j}\right|)^{1/2}$ of the phases ϕ_j of Andronov-Hopf oscillators (c.f. (Kuramoto 1984)), averaged over many simulations with random initial conditions.

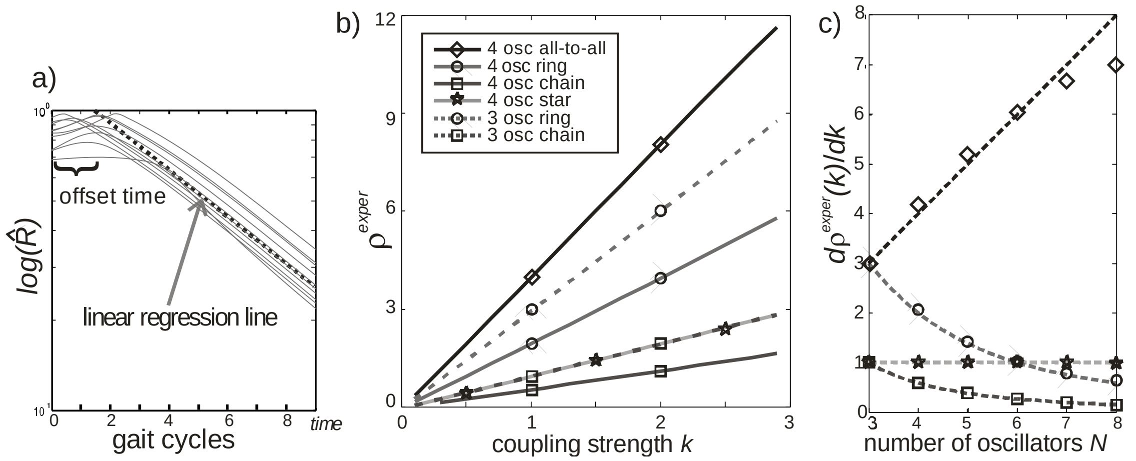

Figure 5. a. Dispersion of the phase of the oscillators, averaged over 100 simulations with random initial conditions, as function of time (gait cycles). After an offset time, during which the dispersion remains relatively constant, it decays exponentially. Convergence rates were estimated by fitting linear function to this decay; **b.** The relationship between convergence rate and coupling strength k for different types of coupling graphs; **c.** Slopes of this relationship as function of the number N of Andronov-Hopf oscillators, comparing simulation results (indicated by the same symbols near to the lines) and derived from the theoretical bounds (Park et al. 2009).

Fig.5a) shows the logarithm of this dispersion measure as a function of time (in gait cycles). After an initial constant interval (offset time), the logarithm of dispersion shows a nearly linear decay with time. From these parts of the curve the rate constants ρ^{exper} were estimated by linear regression. The results for different coupling topologies and for different numbers of avatars are shown in **Fig.5b**) shows the dependency between the coupling strengths *k* and the convergence rate ρ^{exper}, estimated from simulations in the regime of the exponential convergence. As derived from the theoretical bound, the convergence rate varies linearly with the coupling strength. In case of three oscillators the ring coupling is equivalent with all-to-all coupling. **Fig.5c**) shows the slope $d\rho^{exper}(k)/dk$ of this linear relationship as function of *N*, the number of oscillators in the network. We find a close similarity between the theoretically predicted relationship (dashed curves) and the results from the simulation (indicated by the stars). In addition, it is evident that for all-to-all coupling the convergence rate increases with the number of oscillators, while for chain or ring coupling the convergence speed decreases with the number of oscillators (for fixed coupling strength). These results show in particular that the proposed theoretical framework is not only suitable for proving asymptotic stability, but also for guaranteeing the convergence speed of the system dynamics.

5.2 Speed control by variation of step frequency:

The dynamics of this system is given by equations (**2**) and (**3**) for $m_z = 0$. The Jacobian of this system is given by $\mathbf{J}(\boldsymbol{\varphi}) = -m_d \mathbf{L}^d \mathbf{D}_g$, where $(\mathbf{D}_g)_{ii} = g(\phi_i) > 0$ is strictly positive diagonal matrix. The sufficient conditions for exponential contraction are: $\mathbf{J}_s = -m_d \mathbf{B}(\boldsymbol{\varphi}) < 0$, introducing $\mathbf{B}(\boldsymbol{\varphi}) = \mathbf{L}^d \mathbf{D}_g + \mathbf{D}_g (\mathbf{L}^d)^T$.
The exponential contraction conditions are thus satisfied when $\mathbf{B}(\boldsymbol{\varphi}) = \mathbf{L}^d \mathbf{D}_g + \mathbf{D}_g (\mathbf{L}^d)^T > 0$ for any diagonal matrix $\mathbf{D}_g > 0$ and for $\mathbf{L}^d = (\mathbf{L}^d)^T \geq 0$. Matrices $\mathbf{L}^d$ which satisfy this property for $\forall \mathbf{D}_g > 0$ are called *D-stable*. The proof that $\mathbf{B} > 0$ for $\mathbf{L}^d > 0$ and any diagonal $\mathbf{D}_g > 0$ is given by (Bellman 1960, Ch.6, p.94). The proof for real symmetric $\mathbf{L}^d \geq 0$ is following the same line using Hadamard product of two symmetric matrices and exploiting the decomposition $\mathbf{L}^d = \sum_{i=2}^{N} \lambda_i(\mathbf{L}^d) \mathbf{V}_i \mathbf{V}_i^T$, where the $\mathbf{V}_i$ are eigenvectors of $\mathbf{L}^d$, which span its eigenspaces corresponding to distinct eigenvalues $\lambda_i(\mathbf{L}^d)$ ordered in ascending order, and $\lambda_1(\mathbf{L}^d) = 0$ (c.f. Schur 1911, Johnson 1974). For $i \in [2,\ldots,N]$, $\mathbf{V}_i$ are the rows of the matrix $\hat{\mathbf{V}}$, the matrix of projection to the orthogonal complement of the flow-invariant manifold. Using this decomposition, it is straightforward to show that $\mathbf{B}(\boldsymbol{\varphi}) \geq 0$, and that $\mathbf{B}(\boldsymbol{\varphi}) > 0$ unless $\phi_1^* = \ldots = \phi_N^*$. Along the same line, a lower bound for the contraction rate is computed from the projected symmetrized Jacobian $\hat{\mathbf{V}} \mathbf{J}_s(\boldsymbol{\varphi}) \hat{\mathbf{V}}^T = -(m_d / 2) \hat{\mathbf{V}} \mathbf{B}(\boldsymbol{\varphi}) \hat{\mathbf{V}}^T$, resulting in the guaranteed contraction rate $\rho_\varphi = m_d \min_\phi (g(\phi)) \lambda^+_{\mathbf{L}^d}$.
Due to the row diagonal dominance of $\mathbf{L}^d$, for the positive symmetric coupling coefficients $K_{ij} = K_{ji} > 0$ one can show $\mathbf{L}^d \geq 0$ applying Gershgorin's theorem (Horn & Johnson 1985). This last step proves the global stability of system with positive interactions $K_{ij} = K_{ji} > 0$.
In order to suppress oscillatory fluctuations of the positions of the characters the above dynamics can be extended by a low-pass filtering of the character positions: $\dot{\mathbf{z}}(t) = -\alpha \mathbf{z}(t) + G(\phi(t)) + \tilde{\mathbf{c}}$. By introduction of a linear coordinate transformation, bounds for the positive filter constant α can be derived that guarantee globally stable behavior of the system. The dynamics of the system with low-pass filtering is given by

$$\begin{cases} \dot{\boldsymbol{\varphi}}(t) = -m_d \mathbf{L}^d \mathbf{z} \\ \dot{\mathbf{z}} = -\alpha \mathbf{z} + G(\phi) + \tilde{\mathbf{c}} \end{cases}$$

which can be transformed linearly by introduction of the new coordinates $\mathbf{y} = -(m_d / \alpha) \mathbf{L}^d \mathbf{z}$ and $\mathbf{x} = \mathbf{y} + \boldsymbol{\varphi}$:

$$\begin{cases} \dot{\mathbf{x}} = -(m_d / \alpha) \mathbf{L}^d G(\mathbf{x} - \mathbf{y}) \\ \dot{\mathbf{y}} = -\alpha \mathbf{y} - (m_d / \alpha) \mathbf{L}^d G(\mathbf{x} - \mathbf{y}) \end{cases}$$

The Jacobian of this system is $\mathbf{J}(\boldsymbol{\varphi}) = -(m_d / \alpha)\begin{bmatrix} \mathbf{L}^d \mathbf{D}_g & -\mathbf{L}^d \mathbf{D}_g \\ \mathbf{L}^d \mathbf{D}_g & (\alpha^2 / m_d)\mathbf{I} - \mathbf{L}^d \mathbf{D}_g \end{bmatrix}$. Derived from positivity of symmetrized Jacobian, the sufficient stability conditions are $(m_d / \alpha)\mathbf{B}(\boldsymbol{\varphi}) > 0$. As in the unfiltered case, this is condition is fulfilled for $g(\boldsymbol{\varphi}) > 0$, $m_d > 0$, $\alpha > 0$, and $(\mathbf{L}^d)_s \geq 0$, and when additionally: $\alpha^2 / m_d > 1/2 \lambda_{\max}(\mathbf{B})$. The last inequality results in a lower bound for the filter coefficient $\alpha > \sqrt{m_d \min_\phi (g(\phi)) \lambda_{\max}(\mathbf{L}_s^d)}$.

An illustration of these stability bounds if given by the [**Movie**[1]]; that shows convergent behavior of the characters when the contraction condition $(\mathbf{L}^d)_s \geq 0$ is satisfied. [**Movie**[2]] shows the divergent behavior of a group when this condition is violated.

5.3 Stepsize control combined with a control of step phase:

The dynamics is given by equations (**3**) and (**4**) with $m_d = 0$. It can be shown that the dynamics for $\mathbf{z}(t)$ is partially contracting for any external input $\boldsymbol{\varphi}(t)$, if $m_z > 0$, $\boldsymbol{\omega}(t) > 0$ and $(\mathbf{L}^z)_s \geq 0$. This defines a case of a hierarchical system (in form of (**8**)), and the effective relaxation time is determined by equation (**4**). This subsystem is partially contracting, if $k > 0$ and $(\mathbf{L}^\phi)_s \geq 0$. In this case, the effective relaxation time for phase synchronization is given by $\rho_\varphi = k\lambda^+_{\mathbf{L}^\phi}$, where $\lambda^+_{\mathbf{L}^\phi}$ is the smallest non-zero eigenvalue of $(\mathbf{L}^\phi)_s$. This implies that the relaxation time for the distance control is determined by the minimum of ρ_φ and ρ_z, where $\rho_z = m_z \min_\phi (g(\phi)) \lambda^+_{\mathbf{L}^z}$.

Demonstrations of this control dynamics satisfying the contraction conditions are shown in [**Movie**[3]], without control of step phase, and in [**Movie**[4]], with control of step phase.

5.4 Advanced scenarios:

A simulation of a system with stable dynamics with both types of speed control (via stepsize and step frequency) and step phase control is demonstrated in [**Movie**[5]]. Applying the same dynamics, a larger crowd simulated with Horde3d (open-source 3D rendering engine, http://horde3d.org/) is shown in [**Movie**[6]]. In this scenario, the dynamic obstacle avoidance and the control of heading direction were activated in an initial time interval for unsorting of a formation of characters. In a second time interval navigation is deactivated, and speed and position control according to the discussed principles takes over. The development of stability bounds and estimates of relaxation times for such advanced scenarios is the goal of ongoing and future extensions of the proposed approach.

6. CONCLUSION

For the example of a learning-based system for the animation of locomoting groups, we have demonstrated first examples of an application of Contraction Theory for the analysis and the design of stability and convergence properties of collective behaviors in animated crowds. Obviously, future work has to extend this work to more complex scenarios. A generalization of this approach to other problems in character animation,

[1] http://www.uni-tuebingen.de/uni/knv/arl/avi/huma/video1.avi
[2] http://www.uni-tuebingen.de/uni/knv/arl/avi/huma/video2.avi
[3] http://www.uni-tuebingen.de/uni/knv/arl/avi/huma/video3.avi
[4] http://www.uni-tuebingen.de/uni/knv/arl/avi/huma/video4.avi
[5] http://www.uni-tuebingen.de/uni/knv/arl/avi/huma/video5.avi
[6] http://www.uni-tuebingen.de/uni/knv/arl/avi/huma/video6.avi

such as the control of goal-directed behavior and an extension for non-periodic movements seems possible, and such extensions form the core of the planned future work.

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