Analysis and design of the dynamical stability of collective behavior in crowds

Albert Mukovskiy
Section for Computational Sensomotorics, Department of Cognitive Neurology, Hertie Institute for Clinical Brain Research & Centre for Integrative Neuroscience, University Clinic, Tübingen, Germany
albert.mukovskiy@medizin.uni-tuebingen.de

Jean-Jacques E. Slotine
Nonlinear Systems Laboratory, Department of Mechanical Engineering, MIT; Cambridge, MA, USA
jjs@mit.edu

Martin A. Giese
Section for Computational Sensomotorics, Department of Cognitive Neurology, Hertie Institute for Clinical Brain Research & Centre for Integrative Neuroscience, University Clinic, Tübingen, Germany
martin.giese@uni-tuebingen.de

ABSTRACT

The modeling of the dynamics of the collective behavior of multiple characters is a key problem in crowd animation. Collective behavior can be described by the solutions of large-scale nonlinear dynamical systems that describe the dynamical interaction of locomoting characters with highly nonlinear articulation dynamics. The design of the stability properties of such complex multi-component systems has been rarely studied in computer animation. We present an approach for the solution of this problem that is based on Contraction Theory, a novel framework for the analysis of the stability complex nonlinear dynamical systems. Using a learning-based realtime-capable architecture for the animation of crowds, we demonstrate the application of this novel approach for the stability design for the groups of characters that interact in various ways. The underlying dynamics specifies control rules for propagation speed and direction, and for the synchronization of the gait phases. Contraction theory is not only suitable for the derivation of conditions that guarantee global asymptotic stability, but also of minimal convergence rates. Such bounds permit to guarantee the temporal constraints for the order formation in self-organizing interactive crowds.

Keywords: computer animation, crowd animation, coordination, distributed control, stability.

1 INTRODUCTION

Dynamical systems are frequently applied in crowd animation for the simulation of autonomous and collective behavior of many characters [MT01], [TCP06]. Some of this work has been inspired by observations in biology, showing that coordinated behavior of large groups of agents, such as flocks of birds, can be modelled as emergent behavior that arises from the dynamical coupling between interacting agents, without requiring an external central mechanism that ensures coordination [CS07, Cou09], [CDF$^+$01]. Such models can be analyzed by application of methods from nonlinear dynamics [PRK03]. The simulation of collective behavior by self-organization in systems of dynamically coupled agents is interesting because it might reduce the computational costs of traditional computer animation techniques, such as scripting or path planning [TCP06, Rey87]. In addition, the generation of collective behavior by self-organization allows to implement spontaneous adaptation to external perturbations or changes in the system architecture, such as the variation of the number of characters. However, due to the complexity of the models describing individual characters the mathematical analysis of the underlying dynamical systems is typically quite complicated.

In crowd animation, some recent studies have tried to learn interaction rules from the behavior of real human crowds [DH03], [PPS07], [LFCCO09]. Other work has tried to optimize interaction behavior in crowds by exhaustive search of the parameter space exploiting computer simulations by definition of appropriate cost functions (e.g. [HMFB01]). However, most of the existing approaches for the control of group motion in computer graphics have not taken into account the effects of the articulation during locomotion on the control dynamics [PAB07], [NGCL09], [KLLT08]. Consequently, the convergence and stability properties of such dynamical animation systems have rarely been addressed. Distributed control theory has started to study the temporal and spatial self-organization of crowds of agents, and the design of appropriate dynamic interactions, typically assuming rather simple and often even linear agent models (e.g. [SS06], [PLS$^+$07], [SDE95]). However, human-like characters are characterized by highly complex kinematic and even dynamic properties, c.f. [BH97]. Consequently, approaches for a systematic analysis and design of the dynamical properties

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of crowd animation systems are largely lacking. However, such methods seem highly desirable, since they permit one to guarantee desired system properties and to ensure the robustness of the generated behavior under variations of system inputs and the system parameters.

In this paper we introduce Contraction Theory ([LS98], [PS07]) as a framework that makes such stability problems tractable, even for characters with multiple coupled levels of control. Contraction Theory provides a useful tool specifically for modularity-based stability analysis and design [Slo03], [WS05]. This framework is applied to a simple learning-based animation architecture for the real-time synthesis of the movements of interacting characters, which is based on a method that approximates complex human behavior by relatively simple nonlinear dynamical systems [GMP+09], [PMSA09]. Consistent with related approaches in robotics [RI06], [BC89], [GRIL08], [Ijs08], [BRI06], [GTH98], [CS93], this method generates complex movements by the combination of the learned movement primitives [OG06], [GMP+09]. The resulting system architecture is rather simple, making it suitable for a mathematical treatment of dynamical stability properties.

The paper is structured as follows: The structure of the animation system is sketched in section 2. The dynamics underlying navigation control is described in section 3. Subsequently, in section 4 we introduce some basic ideas from Contraction Theory. The major results of our stability analysis and some demos of their applications to the control of crowds are described in section 5, followed by the conclusions.

2 SYSTEM ARCHITECTURE

Our investigation of the collective dynamics of crowds was based on a learning-based animation system, described in details in [GMP+09] (see Fig. 1). By applying anechoic demixing [OG06] to motion capture data, we learned spatio-temporal components. These source components were generated online by nonlinear dynamical systems, Andronov-Hopf oscillators. The mappings σ_j between the stable solutions of the nonlinear oscillators and the required source functions were learned by application of kernel methods [GMP+09]. Each character is modelled by a single limit cycle oscillator, whose solution is mapped by support vector regression (SVR) onto three source signals. These signals were then superimposed with different linear weights w_{ij} and phase delays τ_{ij} in order to generate the joint angle trajectories $\xi_i(t)$ (see Fig. 1). By blending of the mixing weights and the phase delays, intermediate gait styles were generated. This allowed us to simulate specifically walking along paths with different curvatures, changes in step length and walking style. Interactive behavior of multiple agents can be modelled by making the states of the oscillators and the mixing

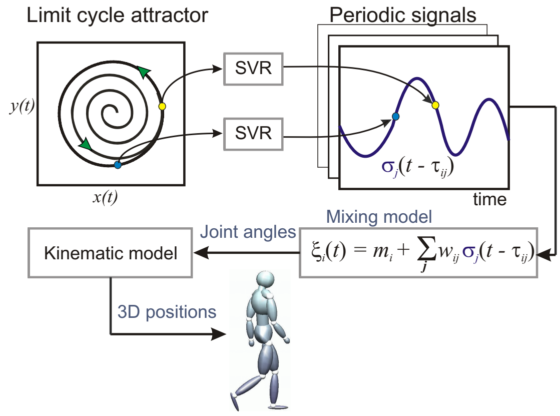

Figure 1: Architecture of the simulation system.

weights dependent on the behavior of the other agents. Such couplings result in a highly nonlinear system dynamics.

The heading direction of the characters was changed by morphing between curved gaits, controlled by a nonlinear navigation dynamics. In the shown applications this dynamics steers the avatars towards goal points that were placed along parallel straight lines. The heading dynamics was given by a nonlinear first-order differential equation (see [GMP+09] for details). Control of heading direction was only active during the the initial stage of the organization of the crowd, resulting in an alignment of the avatars along the parallel straight lines, independent of their initial positions and gait phases. (See Fig. 2 and Fig. 3).

3 CONTROL DYNAMICS

Beyond the control of heading direction, the analyzed scenarios of order formation in a group of characters require the control of the following variables: 1) phase within the step cycle, 2) step length, 3) gait frequency, and 4) heading direction.

The dynamics of each individual character was modelled by an Andronov-Hopf oscillator with constant equilibrium amplitude ($r_i^* = 1$). For appropriate choice of parameters, these nonlinear oscillators have a stable limit cycle that corresponds to a circular trajectory in phase space [AVK87].

In polar coordinates and with the instantaneous eigenfrequency ω this dynamics is given by: $\dot{r}(t) = r(t)\left(1 - r^2(t)\right)$, $\dot{\phi}(t) = \omega$. Control affects the instantaneous eigenfrequency ω of the Andronov-Hopf oscillators and their phases ϕ, while the first equation guarantees that the state stays on the limit cycle ($r(t) = 1, \forall t$).

The position z_i of each character along the parallel paths (see Fig. 2) fulfills the differential equation $\dot{z}_i(t) = \dot{\phi}_i g(\phi_i)$, where the positive function g determines the propagation speed of the character depending on the phase within the gait cycle. This nonlinear

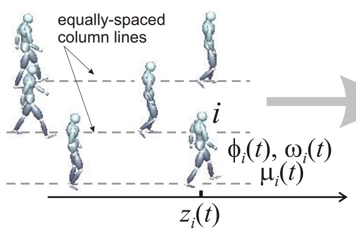

Figure 2: Crowd coordination scenario. Every character i is characterized by its position $z_i(t)$, the phase $\phi_i(t)$ and the instantaneous eigenfrequency $\omega_i(t) = \dot{\phi}_i(t)$ of the corresponding Andronov-Hopf oscillator, and a step-size scaling parameter $\mu_i(t)$.

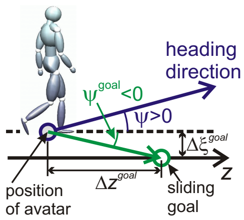

Figure 3: A sliding goal for each avatar was placed on a straight line at fixed distance ahead in z-direction. Heading direction angle: $\psi^{heading}$ and goal direction angle: ψ^{goal}.

function was determined empirically from a kinematic model of character. By integration of this propagation dynamics one obtains $z_i(t) = G(\phi_i(t) + \phi_i^0) + c_i$, with an initial phase shift ϕ_i^0 and some constant c_i depending on the initial position and phase of avatar i, and with the monotonously increasing function $G(\phi_i) = \int_0^{\phi_i} g(\phi) d\phi$, assuming $G(0) = 0$. Three control rules described:

I) Control of step frequency: A simple form of speed control is based on making the frequency of the oscillators $\dot{\phi}_i$ dependent on the behavior of the other characters. Let ω_0 be the equilibrium frequency of the oscillators without interaction. Then a simple controller is defined by the differential equation

$$\dot{\phi}_i(t) = \omega_0 - m_d \sum_{j=1}^{N} K_{ij}[z_i(t) - z_j(t) - d_{ij}] \qquad (1)$$

The constants d_{ij} specify the stable pairwise relative distances in the formed order for each pair (i, j) of characters. The elements of the link adjacency matrix $\mathbf{K}$ are $K_{ij} = 1$ if characters i and j are coupled and zero otherwise. In addition, we assume $K_{ii} = 0$. The constant $m_d > 0$ defines the coupling strength.

With the Laplacian $\mathbf{L}^d$ of the coupling graph, which is defined by $L_{ij}^d = -K_{ij}$ for $i \neq j$ and $L_{ii}^d = \sum_{j=1}^N K_{ij}$, and the constants $c_i = -\sum_{j=1}^N K_{ij} d_{ij}$ the last equation system can be re-written in vector form:

$$\dot{\phi} = \omega_0 \mathbf{1} - m_d(\mathbf{L}^d G(\phi + \phi^0) + \mathbf{c}) \qquad (2)$$

II) Control of step length: Step length was varied by morphing between gaits with short and long steps. A

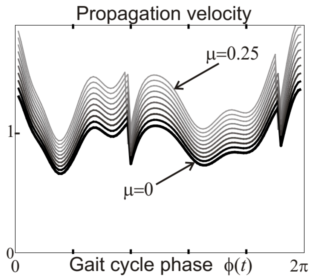

Figure 4: Propagation velocity for 10 different values the of step length morphing parameter $\mu = [0 \ldots 0.25]$ dependent on gait cycle phase $\phi(t)$ and $\omega(t) = 1$. The vertical axis is scaled in order to make all average velocities equal to one for $\mu = 0$ (lowest thick line). This empirical estimates are well approximated by $(1 + \mu)g(\phi(t))$.

detailed analysis shows that the influence of step length on the propagation could be well captured by simple linear rescaling. If the propagation velocity of characters i is $v_i(t) = \dot{z}_i(t) = \dot{\phi}_i(t)g(\phi_i(t)) = \omega_i(t)g(\phi_i(t))$ for the normal step size, then the velocity for modified step size was well approximated by $v_i(t) = \dot{z}_i(t) = (1 + \mu_i)\omega_i(t)g(\phi_i(t))$ with the morphing parameter μ_i. The range of morphing parameters was restricted to the interval $-0.5 < \mu_i < 0.5$, where this linear scaling law was fulfilled with high accuracy. The empirically estimated propagation velocity in heading direction, dependent on gait phase, is shown in Fig.4 for different values of the step length morphing parameter μ_i. Using the same notations as in equation (1), this motivates the definition of the following dynamics that models the influence of the step length control on the propagation speed:

$$\dot{\mathbf{z}} = \omega g(\phi + \phi^0)(1 - m_z(\mathbf{L}^z \mathbf{z} + \mathbf{c})) \qquad (3)$$

In this equation $\mathbf{L}^z$ signifies the Laplacian of the relevant coupling graph, and m_z the strength of the coupling. For uncoupled characters ($m_z = 0$) this equation is consistent with the the definition of propagation speed that was given before.

III) Control of step phase: By defining separate controls for step length and step frequency it becomes possible to dissociate the control of position and step phase of the characters. Specifically, it is interesting to introduce a controller that results in phase synchronization between different characters. This can be achieved by addition of a simple linear coupling term to equation (1), written in vector form:

$$\dot{\phi} = \omega_0 \mathbf{1} - m_d(\mathbf{L}^d G(\phi + \phi^0) + \mathbf{c}) - k\mathbf{L}^\phi \phi \qquad (4)$$

with $k > 0$ and the Laplacian $\mathbf{L}^\phi$. (All sums or differences of angular variables were computed by modulo 2π).

IV) Control of heading direction:

The heading directions of the characters were controlled by a navigation dynamics that steers the avatars towards goal points, which were placed along parallel straight lines in front of the avatars (2). The heading dynamics was given by a nonlinear differential equation, independently for every character [GMP+09]:

$$\dot{\psi}_i = \sin(\psi_i^{goal} - \psi_i) \tag{5}$$

where $\psi_i^{goal} = \arctan(\Delta\xi_i^{goal}/\Delta z_i^{goal})$, $\Delta\xi_i^{goal}$ is the distance to the goal line in the direction orthogonal to the propagation direction, while Δz_i^{goal} is constant, (see Fig. 3). The morphing weight that controls the mixture of walking with left and right turning was proportional to $\dot{\psi}_i(t)$. For the mathematical stability analysis presented in the following, we neglected the influence of the dynamics of the control of heading direction, focusing on the order formation scenarios when the agents' heading directions are already aligned, when they walk along parallel straight lines towards sliding goal points. In this case, the positions of the agents can be described by a single position variable $z(t)$. An extension of the developed analysis framework including the control of the heading direction is in progress.

The mathematical results derived in the following sections apply to subsystems of the complete system dynamics that is given by equations (3) and (4). In addition, simulations are presented for the full system dynamics.

4 CONTRACTION THEORY

Dynamical systems describing the behavior of autonomous agents are essentially nonlinear. In contrast to the linear dynamical systems, a major difficulty of the analysis of stability properties of nonlinear is that the stability properties of parts usually do not transfer to composite systems. Contraction Theory [LS98] provides a general method for the analysis of essentially nonlinear systems, which permits such a transfer, making it suitable for the analysis of complex systems with many components. Contraction Theory characterizes the system stability by the behavior of the differences between solutions with different initial conditions. If these differences vanish exponentially over time, all solutions converge towards a single trajectory, independent from the initial states. In this case, the system is called *globally asymptotically stable*. For a general dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{6}$$

assume that $\mathbf{x}(t)$ is one solution of the system, and $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) + \delta\mathbf{x}(t)$ a neighboring one with a different initial condition. The function $\delta\mathbf{x}(t)$ is also called *virtual displacement*. With the Jacobian of the system $\mathbf{J}(\mathbf{x},t) = \frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial \mathbf{x}}$ it can be shown [LS98] that any nonzero virtual displacement decays exponentially to zero over time if the symmetric part of the Jacobian $\mathbf{J}_s = (\mathbf{J} + \mathbf{J}^T)/2$ is uniformly negative definite, denoted as $\mathbf{J}_s < 0$. This implies that it has only negative eigenvalues for all relevant state vectors $\mathbf{x}$. In this case, it can be shown that the norm of the virtual displacement decays at least exponentially to zero, for $t \to \infty$. If the virtual displacement is small enough, then

$$\dot{\delta\mathbf{x}}(t) = \mathbf{J}(\mathbf{x},t)\delta\mathbf{x}(t)$$

implying through $\frac{d}{dt}||\delta\mathbf{x}(t)||^2 = 2\delta\mathbf{x}^T(t)\mathbf{J}_s(\mathbf{x},t)\delta\mathbf{x}$ the inequality: $||\delta\mathbf{x}(t)|| \leq ||\delta\mathbf{x}(0)||\, e^{\int_0^t \lambda_{\max}(\mathbf{J}_s(\mathbf{x},s))\,ds}$. This implies that the virtual displacements decay with a *convergence rate* (inverse timescale) that is bounded from below by the quantity $\rho_c = -\sup_{\mathbf{x},t} \lambda_{\max}(\mathbf{J}_s(\mathbf{x},t))$, where $\lambda_{\max}(.)$ signifies the largest eigenvalue. With $\rho_c > 0$ all trajectories converge to a single solution exponentially in time [LS98].

Contraction analysis can be applied also to hierarchically coupled systems [LS98]. Consider a composite dynamical system with two components, where the dynamics of the first subsystem is not influenced by the dynamics of the second one. Such system is called *hierarchically coupled*. The composite dynamical system:

$$\frac{d}{dt}\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1(\mathbf{x}_1) \\ \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2) \end{pmatrix} \tag{7}$$

results in the Jacobian:

$$\mathbf{F} = \begin{pmatrix} \frac{\partial \mathbf{f}_1(\mathbf{x}_1)}{\partial \mathbf{x}_1} & 0 \\ \frac{\partial \mathbf{f}_2(\mathbf{x}_1,\mathbf{x}_2)}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{f}_2(\mathbf{x}_1,\mathbf{x}_2)}{\partial \mathbf{x}_2} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_{11} & 0 \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{pmatrix} \tag{8}$$

Consider then the smooth dynamics of virtual displacements: $\frac{d}{dt}\begin{pmatrix} \delta\mathbf{x}_1 \\ \delta\mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_{11} & 0 \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{pmatrix}\begin{pmatrix} \delta\mathbf{x}_1 \\ \delta\mathbf{x}_2 \end{pmatrix}$, where $\mathbf{F}_{21}$ is bounded. The first subsystem does not depend on the second, so that $\delta\mathbf{x}_1$ exponentially converges to 0 if $(\mathbf{F}_{11})_s < 0$. Then, $\mathbf{F}_{21}\delta\mathbf{x}_1$ is an exponentially decaying disturbance for the second subsystem. In this case, (see [LS98] for details of proof), uniformly negative definite $\mathbf{F}_{22}$ implies exponential convergence of $\delta\mathbf{x}_2$ to an exponentially decaying ball. The whole system is then globally exponentially convergent to a single trajectory.

Many systems are not contracting with respect to all dimensions of the state space, but show convergence with respect to a subset of dimensions. Such behavior can be mathematically characterized by *partial contraction* [WS05], [PMSA09]. The underlying idea is to construct an auxiliary system that is contracting with respect to a subset of the arguments of the function $\mathbf{f}$. The major result is the following:

Theorem 1 *Consider a nonlinear system of the form*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t) \tag{9}$$

and assume the existence of auxiliary system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, t) \tag{10}$$

that is contracting with respect to $\mathbf{y}$ *uniformly for all relevant* $\mathbf{x}$*. If a particular solution of this auxiliary system verifies a specific smooth property, then trajectories of the original system (9) verify this property exponentially. The original system is then said to be partially contracting. [WS05].*

A 'smooth property' is a property of the solution that depends smoothly on space and time, such as convergence against a particular solution or a properly defined distance to a subspace in phase space. The proof of the theorem is immediate noticing that the observer-like system (10) has $\mathbf{y}(t) = \mathbf{x}(t)$ for all $t \geq 0$ as a particular solution. Since all trajectories of the $\mathbf{y}$-system converge exponentially to a single trajectory, this implies that also the trajectory $\mathbf{x}(t)$ verifies this specific property with exponential convergence.

It is thus sufficient to show that the auxiliary system is contracting in order to prove the convergence to a subspace. Let us assume that system has a flow-invariant *linear subspace* $\mathscr{M}$, which is defined by the property that trajectories starting in this space always remain in it for arbitrary times ($\forall t : \mathbf{f}(\mathscr{M},t) \subset \mathscr{M}$). If matrix $\mathbf{V}$ is an orthonormal projection onto $\mathscr{M}^{\perp}$, then sufficient condition for global exponential convergence to $\mathscr{M}$ is:

$$\mathbf{V}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{s} \mathbf{V}^T < \mathbf{0}, \tag{11}$$

where smaller sign indicates that this matrix is negative definite (see [PS07, PMSA09]).

5 RESULTS

We derived contraction bounds for three scenarios that correspond to control dynamics with increasing levels of complexity.

1) Control of step phase without position control: The simplest case is a control of the phase within the step cycle of the walkers without simultaneous control of the position of the characters. Such simple control already permits to simulate interesting behaviors, such as soldiers synchronizing their step phases [PMSA09], [**Demo**[1]]. The underlying dynamics is given by (4) with $m_d = 0$. For N identical dynamical systems, with symmetric identical coupling gains k this dynamics can be written

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + k \sum_{j \in \mathscr{N}_i} (\mathbf{x}_j - \mathbf{x}_i), \quad \forall i = 1, \ldots, N \tag{12}$$

where $\mathscr{N}_i$ defines the index set specifying the neighborhood in the coupling graph, i.e. the other subsystems or characters that are coupled with character i.

This type of symmetric coupling, where the interaction forces between subsystems depend only on the differences of the phase variables is called *diffusive coupling*. In this case, the *Laplacian matrix* of the coupling scheme is given by $\mathbf{L} = \mathbf{L}_G \bigotimes I_p$, where p is the dimensionality of the individual sub-systems, and where $\bigotimes$ signifies the *Kronecker product*. The Laplacian of the coupling graph is the matrix $\mathbf{L}_G$. The system then can be rewritten compactly as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) - k\mathbf{L}\mathbf{x}$ with the concatenated phase variable $\mathbf{x} = [\mathbf{x}_1^T, ..., \mathbf{x}_n^T]^T$. The Jacobian of this system is $\mathbf{J}(\mathbf{x},t) = \mathbf{D}(\mathbf{x},t) - k\mathbf{L}$, where the block-diagonal matrix $\mathbf{D}(\mathbf{x},t)$ has the Jacobians of the uncoupled components $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_i,t)$ as entries.

The dynamics has a flow-invariant linear subspace $\mathscr{M}$ that contains the particular solution $\mathbf{x}_1^* = \cdots = \mathbf{x}_n^*$. For this solution all state variables $\mathbf{x}_i$ are identical and thus in synchrony. In addition, for this solution the coupling term in equation (12) vanishes, so that the form of the solution is identical with the solution of the uncoupled systems $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i)$. If $\mathbf{V}$ is a projection matrix onto the subspace $\mathscr{M}^{\perp}$, then, according to (11), the sufficient contraction condition for convergence toward $\mathscr{M}$ is given by $\mathbf{V}(\mathbf{D}(\mathbf{x},t) - k\mathbf{L})_s \mathbf{V}^T < \mathbf{0}$, [PMSA09]. This implies

$$\lambda_{\min}\left(\mathbf{V}(k\mathbf{L})_s \mathbf{V}^T\right) = k\lambda_{\mathbf{L}}^{+} > \sup_{\mathbf{x},t} \lambda_{\max}\left(\mathbf{D}_s\right)$$

with $\lambda_{\mathbf{L}}^{+}$ being the smallest non-zero eigenvalue of symmetrical part of the Laplacian $\mathbf{L}_s$. The maximal eigenvalue for the individual oscillator is $\sup_{\mathbf{x},t} \lambda_{\max}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x},t)\right)$. The sufficient condition for global stability of the overall system is given by $k > \sup_{\mathbf{x},t} \lambda_{\max}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x},t)\right) / \lambda_{\mathbf{L}}^{+}$. This implies the following minimum convergence rate: $\rho_c = -\sup_{\mathbf{x},t} \lambda_{\max}(\mathbf{V}(\mathbf{D}(\mathbf{x},t) - \mathbf{L})_s \mathbf{V}^T)$.

For the special case of (4) with $m_d = 0$ this implies the sufficient contraction conditions $k > 0$ and $(\mathbf{L}^{\phi})_s \geq 0$.

Different topologies of the coupling graphs result in different stability conditions, since for example $\lambda_{\mathbf{L}}^{+} = 2(1 - \cos(2\pi/N))$ for symmetric ring coupling, and $\lambda_{\mathbf{L}}^{+} = N$ for all-to-all coupling. (N is the number of avatars.) See [WS05] and [PMSA09] for details.

2) Speed control by variation of step frequency: The dynamics of this system is given by equations (2) and (3) for $m_z = 0$. Assuming arbitrary initial distances and phase offsets for different propagating characters, implying $G(\phi_i^0) = c_i$, $c_i \neq c_j$, for $i \neq j$, we redefine d_{ij} as $d_{ij} - (c_i - c_j)$ in (1), and accordingly redefine $\mathbf{c}$ in (2). Assuming this control dynamics, and two avatars i and j that follow a leading avatar, their phase trajectories converge to a single unique trajectory only if $c_i = c_j$. This is a consequence of the one-to-one correspondence between gait phase and position of the avatar that is given by equation (2). In all other cases the trajectories of the followers converge to one-dimensional, but distinct, attractors in phase-position space that are uniquely defined by c_i. These attractors correspond to a behavior where the follower's position oscillates around the position of the leader.

[1] www.uni-tuebingen.de/uni/knv/arl/avi/wscg/video0.avi

For the analysis of contraction properties we regard an auxiliary system obtained from (2) by keeping the terms which are only dependent on ϕ: $\dot{\phi} = -m_d \mathbf{L}^d G(\phi + \phi^0)$. The symmetrized Jacobian of this system projected to the orthogonal compliment of flow-invariant linear subspace $\phi_1^* + \phi_1^0 = \ldots = \phi_N^* + \phi_N^0$ determines whether this system is partially contracting. By virtue of a linear change of variables the study of the contraction properties of this system is equivalent to study the contraction properties of the dynamical system $\dot{\phi} = -m_d \mathbf{L}^d G(\phi)$ on trajectories converging towards its flow-invariant manifold, the linear subspace of $\phi_1^* = \ldots = \phi_N^*$.

In order to derive an asymptotic stability condition, we consider the following auxiliary system, corresponding to a part of (2): $\dot{\phi} = -m_d \mathbf{L}^d G(\phi)$. The Jacobian of this system is given by $\mathbf{J} = -m_d \mathbf{L}^d \mathbf{D}_g$, where $(\mathbf{D}_g)_{ii} = g(\phi_i) = G'(\phi_i) > 0$ is a strictly positive diagonal matrix. Exploiting diagonal stability theory [Per69], it is straightforward to demonstrate that the auxiliary system is globally asymptotically stable and its state converges to an attractor with $\phi_1^* = \ldots = \phi_N^*$ for any initial condition assuming $(\mathbf{L}^d)_s \geq 0$ and $m_d > 0$. The sufficient conditions for asymptotic stability are satisfied for all types of symmetric diffusive couplings with positive coupling strength. For the case of asymmetric coupling graphs with more general structure including negative feedback links some results on asymptotic stability have been provided in [SA08].

The sufficient conditions for (exponential) partial contraction towards flow-invariant subspace are, (see (11)): $\mathbf{V}\mathbf{J}_s(\phi)\mathbf{V}^T = -m_d \mathbf{V}\mathbf{B}(\phi)\mathbf{V}^T < 0$, introducing $\mathbf{B}(\phi) = \mathbf{L}^d \mathbf{D}_g + \mathbf{D}_g (\mathbf{L}^d)^T$ and $\mathbf{V}$ signifying the projection matrix onto the orthogonal complement of the flow-invariant linear subspace. For diffusive coupling with symmetric Laplacian the linear flow-invariant manifold $\phi_1^* = \ldots = \phi_N^*$ is also the null-space of the Laplacian. In this case, the eigenvectors of the Laplacian that correspond to positive eigenvalues can be used to construct the projection matrix $\mathbf{V}$. For $m_d > 0$ the contraction conditions are thus satisfied if $\mathbf{V}\mathbf{B}(\phi)\mathbf{V}^T = \mathbf{V}(\mathbf{L}^d \mathbf{D}_g + \mathbf{D}_g (\mathbf{L}^d)^T)\mathbf{V}^T > 0$ for any diagonal matrix $\mathbf{D}_g > 0$.

Next we prove the exponential contraction conditions for the particular case of symmetrical all-to-all coupling. In this case $\mathbf{L}^d = N\mathbf{I} - \mathbf{1}\mathbf{1}^T \geq 0$, where $\mathbf{I}$ is identity matrix of size N. Since $\mathbf{V}\mathbf{1} = \mathbf{1}^T \mathbf{V}^T = 0$, we obtain $\frac{1}{2}\mathbf{V}(\mathbf{L}^d \mathbf{D}_g + \mathbf{D}_g (\mathbf{L}^d)^T)\mathbf{V}^T = N\mathbf{V}(\mathbf{D}_g)\mathbf{V}^T > 0$ for $\mathbf{D}_g > 0$. A lower bound for the contraction rate is computed from the projected symmetrized Jacobian $\mathbf{V}\mathbf{J}_s(\phi)\mathbf{V}^T = -\frac{m_d}{2}\mathbf{V}\mathbf{B}(\phi)\mathbf{V}^T$. Contraction theory also permits to compute the guaranteed contraction rate $\rho_{min} = m_d \min_\phi (g(\phi))\lambda^+_{\mathbf{L}^d}$, with $\lambda^+_{\mathbf{L}^d} = N$ for all-to-all symmetric coupling.

For a general symmetric couplings with positive links (with equal coupling strength $m_d > 0$) we obtain the sufficient contraction condition as: $\lambda^+_{min}(\mathbf{L}^d)/\lambda^+_{max}(\mathbf{L}^d) > \max_\phi(|g(\phi) - mean(g(\phi))|)/mean(g(\phi))$, where mean value of $g(\phi)$ over the gait cycle period T is: $mean(g(\phi)) = {}^1/_T \int_0^T g(\phi)d\phi$. This condition is derived from the fact that for symmetric (positive) matrices M_1 and M_2 for $M_1 - M_2 > 0$ it is sufficient to satisfy $M_1 > M_2$ (the last means $\lambda_{min}(M_1) > \lambda_{max}(M_2)$). This sufficient condition put the constraints on admissible topologies of the coupling scheme dependent on the smoothness of gait velocity function $g(\phi)$. Alternatively, it is possible to introduce low-pass filtering of the forward kinematics of walking characters in order to increase the smoothness of $g(\phi)$.

An illustration of these stability bounds if given by the [**Demo**[2]]; that shows convergent behavior of the characters when the contraction condition $m_d > 0, (\mathbf{L}^d)_s \geq 0$ is satisfied for all-to-all coupling. [**Demo**[3]] shows the divergent behavior of a group when this condition is violated when $m_d < 0$.

The same proof can be extended for **nonlinear control rules**. In this case the eigenfrequency is given by a nonlinear modification of the control rule in eq. (1), for character i coupled to character j as: $\omega_i = \omega_0 + m_d h(z_j - z_i + d_{ij})$, where the saturating nonlinear function h could be given, for example by $h(z) = 1/[1 + \exp(-\gamma z)]$ with $\gamma > 0$. This nonlinear function limits the range of admissible speeds for the controller. Using the same notations as above, the dynamics of a single follower that follows a leader at position $P(t)$ is given by: $\dot{\phi}(t) = \omega_0 + m_d h(P(t) - G(\phi(t)) + c)$. The Jacobian of this dynamics $\mathbf{J}_s = -m_d h' g(\phi) < 0$ is negative, which follows from $m_d > 0$, $g(\phi) > 0$ and taking into account $h'(z) = dh(z)/dz > 0, \forall z$, what guarantees contraction.

Again this dynamics can be extended for N avatars, resulting in the nonlinear differential equation system: $\dot{\phi}_i(t) = \omega_0 - m_d \sum_{j=1}^N K_{ij} h(G(\phi_i) - G(\phi_j) + d_{ij}), \forall i$. The Jacobian of the system is: $\mathbf{J}(\phi) = -m_d \mathbf{L}^d(\phi)\mathbf{D}_g$, where $\mathbf{L}^d_{ij}(\phi) = -K_{ij} h'(G(\phi_i) - G(\phi_j) + d_{ij})$, $\mathbf{L}^d_{ii}(\phi) = \sum_{j \neq i}^N K_{ij} h'(G(\phi_i) - G(\phi_j) + d_{ij})$, $d_{ii} = 0$, ($\mathbf{D}_g$ is defined as before). Furthermore, the even function $h'(z) > 0$ implies that the Laplacian $\mathbf{L}^d(\phi)$ is symmetric diagonally dominant and it stays positive semidefinite for any positive $K_{ij} > 0$, by Gershgorin's Theorem [HJ85]. This implies that the system is asymptotically stable, its solutions converging to an attractor. The analysis of exponential convergence requires further steps that exceed the scope of this paper.

3) Stepsize control combined with a control of step phase: The dynamics is given by equations (3) and (4) with $m_d = 0$. This dynamics defines a hier-

[2] www.uni-tuebingen.de/uni/knv/arl/avi/wscg/video1.avi
[3] www.uni-tuebingen.de/uni/knv/arl/avi/wscg/video2.avi

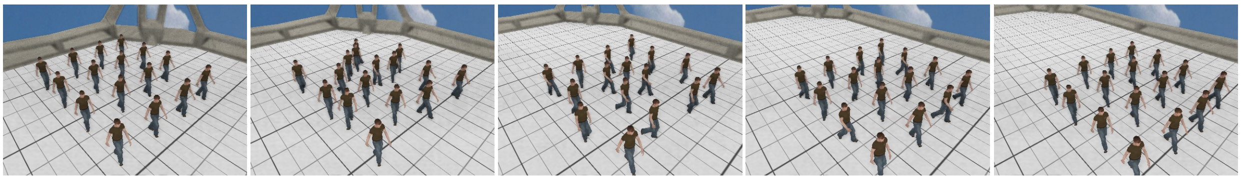

Figure 5: Self-organized reordering of a crowd with 16 characters. Control dynamics affects direction, distance and gait phase. See [**Demo**[7]].

archically coupled nonlinear system (see (7), Section 4), which is difficult to analyze with classical methods [LS98]. The dynamics for $\mathbf{z}(t)$ given by equation (3) is partially contracting in case of all-to-all coupling for any bounded external input $\phi(t)$, if $m_z > 0$, $\mathbf{L}^z \geq 0$ and $\omega(t) > 0$. These sufficient contraction conditions can be derived from the requirement of the positive-definiteness of the symmetrized Jacobian applying a similar technique as above. The Jacobian of this subsystem is $\mathbf{J}(\phi,\omega) = -m_z\mathbf{D}^z_g(\phi,\omega)\mathbf{L}^z$, with the diagonal matrix $(\mathbf{D}^z_g(\phi,\omega))_{ii} = \omega_i g(\phi_i + \phi_i^0) > 0$ that is positive definite since $g(\phi) > 0$ and $\omega > 0$. This subsystem is (exponentially) contracting and its relaxation rate is determined by $\rho_z = m_z \min_\phi (g(\phi))\lambda^+_{\mathbf{L}^z}$ (in the case of all-to-all coupling) for any input from the dynamics of $\phi(t)$ eq. (4). The last dynamics is contracting when $(\mathbf{L}^\phi)_s \geq 0$ and its relaxation rate is $\rho_\phi = k\lambda^+_{\mathbf{L}^\phi}$, where $\lambda^+_{\mathbf{L}^\phi}$ is the smallest non-zero eigenvalue of $(\mathbf{L}^\phi)_s$. The effective relaxation time of the overall dynamics is thus determined by the minimum of the contraction rates ρ_ϕ and ρ_z.

Demonstrations of this control dynamics satisfying the contraction conditions are shown in [**Demo**[4]], without control of step phase, and in [**Demo**[5]], with control of step phase.

4) Advanced scenarios: A simulation of a system with stable dynamics with both types of speed control (via step size and step frequency) and step phase control is shown in [**Demo**[6]]. Using the same dynamics, a larger crowd of 16 avatars simulated with the open-source animation engine Horde3d [Sch09] is shown in [**Demo**[7]]. In this scenario, dynamic obstacle avoidance and control of heading direction were activated in an initial time interval for unsorting of a formation of avatars. In a second time interval navigation is deactivated, and speed and position control according to the discussed principles takes over. [**Demo**[8]] demonstrates a large synchronizing crowd with 36 avatars without initial reordering. [**Demo**[9]] shows the divergence dynamics of the crowd from previous example, when negative distance-to-eigenfrequency coupling strength used ($m_d < 0$). Two videos [**Demo**[10]] and [**Demo**[11]] show convergence dynamics of the crowd of 49 avatars for two different values of the strength of the distance-to-step size coupling (slow and fast). The coupling strength for the phase coupling dynamics is constant for this example. The development of stability bounds and estimates of relaxation times for even more advanced scenarios including multiple control levels of navigation is the goal of ongoing work.

6 CONCLUSION

For the example of a learning-based system for the animation of locomoting groups, we have demonstrated first applications of Contraction Theory for the analysis and the design of stability and convergence properties of collective behaviors in animated crowds. The dynamics of the collective behavior of animated crowds is highly nonlinear and prevents the stability analysis using classical approaches. Opposed to these approaches, Contraction theory allows to transfer stability results from the components to composite systems. Future work has to extend this approach to more complex scenarios, especially including non-periodic movements.

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[4] www.uni-tuebingen.de/uni/knv/arl/avi/wscg/video3.avi
[5] www.uni-tuebingen.de/uni/knv/arl/avi/wscg/video4.avi
[6] www.uni-tuebingen.de/uni/knv/arl/avi/wscg/video5.avi
[7] www.uni-tuebingen.de/uni/knv/arl/avi/wscg/video6.avi
[8] www.uni-tuebingen.de/uni/knv/arl/avi/wscg/video7.avi
[9] www.uni-tuebingen.de/uni/knv/arl/avi/wscg/video8.avi

[10] www.uni-tuebingen.de/uni/knv/arl/avi/wscg/video9.avi
[11] www.uni-tuebingen.de/uni/knv/arl/avi/wscg/video10.avi

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