Analysis of the global dynamical stability of crowd navigation applying Contraction Theory

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Abstract

The modelling of the collective behavior of many characters is an important problem in crowd animation. Such behaviors can be described by solutions of large-scale nonlinear dynamical systems, built from multiple interacting components. The design of stability properties of such multi-component systems has been rarely studied in computer animation. We propose an approach for the solution of this problem that is based on Contraction Theory, a novel method for the stability analysis of complex nonlinear dynamical systems. This novel approach was applied to derive bounds that guarantee the global asymptotic stability and convergence rates for navigating crowds of characters with simultaneous control of step frequency and length.

Keywords: computer animation, convergence, stability, coordination, crowd animation

1 Introduction

Dynamical systems are frequently applied in crowd animation for the simulation of autonomous and collective behavior of many characters [1]. Some of this work has been inspired by observations in biology, showing that coordinated behavior of large groups of agents, such as flocks of birds, can be modelled as emergent behavior arising from the dynamical coupling between interacting agents, without requiring an external central mechanism ensuring coordination [2, 3]. Such models can be analyzed by application of methods from nonlinear dynamics [4]. However, this analysis is complicated because the underlying nonlinear agent dynamics is typically highly nonlinear, making a systematic treatment of stability properties often infeasible. Consequently, the convergence and stability properties of such dynamical systems have been rarely addressed in the context of computer animation. This paper introduces Contraction Theory [5] as novel framework for the analysis and design of the convergence properties of navigating avatars during self-organized order formation.

2 System architecture

Our investigation of the collective dynamics of crowds was based on a learning-based animation system [6]. Based on motion capture data we learned spatio-temporal components of sets of different gait types; applying an algorithm for translation-invariant blind source separation [7]. The obtained source components were generated by nonlinear dynamical systems. We

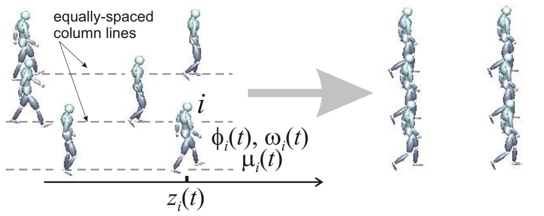

Figure 1: The crowd coordination task setup. Every avatar i is characterized by its position $z_i(t)$, the phase $\phi_i(t)$ and the instantaneous eigenfrequency $\omega_i(t) = \dot{\phi}_i(t)$ of Andronov-Hopf oscillator, and the step-size scaling parameter $\mu_i(t)$.

learned the mapping between the stable solution of a nonlinear oscillator and the required source functions by application of kernel methods. Each character is modelled by a single Andronov-Hopf oscillator whose solution is mapped onto three source signals. These signals were superimposed with different linear weights and phase delays in order to generate the joint angle trajectories. By blending of the mixing weights and the phase delays, intermediate gait styles were generated. This allowed us to simulate specifically walking along paths with different curvatures and changes in step length. Interactive behavior of multiple characters can be modelled by making the states of the oscillators and the mixing weights dependent on the behavior of the other characters. Such coupling results in a highly nonlinear complex system dynamics.

3 Control dynamics

For the simulations presented in this abstract we controlled the following variables: 1) phase of the step cycle, 2) step length, and 3) heading direction. The control of heading direction was accomplished with a standard approach from robotics [6]. It ensured that the characters were locomoting on parallel straight lines, see Fig. 1. The influence of this direction control was neglected in the analysis of the system dynamics as presented in this paper, corresponding to the assumption that the characters deviate only by a negligible amount from the prescribed parallel paths.

Each character was modelled by a Hopf oscillator with constant equilibrium amplitude ($r_i^* = 1$). Control acted on the phases ϕ_i of these oscillators. The position z_i of a character along the parallel paths (see Fig. 1) fulfills the differential equation $\dot{z}_i(t) = \dot{\phi}_i g(\phi_i)$, where the positive function g determines the propagation speed of the character depending on the phase in the gait cycle. This nonlinear function was determined from the kinematics of the character.

I) Control of step frequency: A simple form of speed control is based on making the frequency of the oscillators $\dot{\phi}_i$ dependent on the behavior of the other characters. Let ω_0 be the equilibrium frequency of the oscillators without interaction, then a simple controller is defined by the differential equation

$$\dot{\phi}_i(t) = \omega_0 - m_d \sum_{j=1}^{N} K_{ij}[G(\phi_i(t)) - G(\phi_j(t)) - d_{ij}] \tag{1}$$

with $G(\phi_i) = \int_0^{\phi_i} g(\phi)d\phi = z_i$. The constants d_{ij} define the stable pairwise relative distances in formed order for each pair (i, j) of characters. The elements of the link adjacency matrix $\mathbf{K}$ are $K_{ij} = 1$ if characters i and j are coupled and zero otherwise. In addition, we assume $K_{ii} = 0$. The constant $m_d > 0$ defines the coupling strength.

With the Laplacian $\mathbf{L}^d$ of the coupling graph, which is defined by $L_{ij}^d = -K_{ij}$ for $i \neq j$ and $L_{ii}^d = \sum_{j=1}^N K_{ij}$, and constants $c_i = -\sum_{j=1}^N K_{ij} d_{ij}$ the last equation system can be written in vector form:

$$\dot{\boldsymbol{\phi}} = \omega_0 \mathbf{1} - m_d(\mathbf{L}^d G(\boldsymbol{\phi}) + \mathbf{c}) \tag{2}$$

II) Control of step length: Step length was varied by morphing between gaits with short and long steps. A detailed analysis shows that the influence of step length on the propagation speed is well captured by simple linear rescaling. Using the same notations as in equation (1), this motivates the definition of the following dynamics that models the influence of step length control on the propagation speed:

$$\dot{\mathbf{z}}(t) = \dot{\boldsymbol{\phi}}(t) g(\boldsymbol{\phi}(t))(1 - m_z(\mathbf{L}^z \mathbf{z}(t) + \mathbf{c})) \tag{3}$$

In this equation $\mathbf{L}^z$ signifies the Laplacian of the relevant coupling graph, and m_z the strength of the coupling. For uncoupled characters ($m_z =$

0) this equation is consistent with the the definition of propagation speed that was given before.

III) Control of step phase: By defining separate controls for step length and step frequency it becomes possible to dissociate the control of position and step phase of the characters. Specifically, it is interesting to introduce a controller that results in phase synchronization between different characters. This can be achieved by addition of a simple linear coupling term to equation (1)

$$\dot{\phi} = \omega_0 \mathbf{1} - m_d(\mathbf{L}^d G(\boldsymbol{\phi}) + \mathbf{c}) - k\mathbf{L}^{\phi}\boldsymbol{\phi}(t) \quad (4)$$

with $k > 0$ and the Laplacian $\mathbf{L}^{\phi}$.

The following mathematical results are derived for subsystems of the complete system dynamics that is given by equations (3) and (4). In addition, simulations are presented with the full system dynamics.

4 Contraction Theory

Dynamical systems describing the behavior of autonomous characters are essentially nonlinear. In contrast to the linear dynamical systems, a major difficulty of the analysis of stability properties of nonlinear is that the stability properties of parts usually do not transfer to composite systems. Contraction theory [5] provides a general method for the analysis of essentially nonlinear systems, which permits such a transfer, making it suitable for the analysis of complex systems with many components. Contraction theory characterizes the system stability by the behavior of the differences between solutions with different initial conditions. If these differences vanish exponentially over time, and its solution converges towards a single trajectory, independent from the initial states, the system is called *globally asymptotically stable*. For a general dynamical system of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ assume that $\mathbf{x}(t)$ is one solution of the system and $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) + \delta\mathbf{x}(t)$ a neighboring one with a different initial condition. The function $\delta\mathbf{x}(t)$ is also called *virtual displacement*. With the Jacobian of the system $\mathbf{J}(\mathbf{x}, t) = \frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial \mathbf{x}}$ it can be shown [5] that that any nonzero virtual displacement decays exponentially to zero over time if the symmetric part of the Jacobian $\mathbf{J}_s = (\mathbf{J} + \mathbf{J}^T)/2$ is uniformly negative definite, i.e. has negative eigenvalues for all relevant state vectors $\mathbf{x}$. In this case, it can be shown that the norm of the virtual displacement decays at least exponentially to zero, according to the inequality: $||\delta\mathbf{x}(t)|| \leq ||\delta\mathbf{x}(0)||\ e^{\int_0^t \lambda_{\max}(\mathbf{J}_s(\mathbf{x},s))\,ds}$. This decay occurs with a *convergence rate* (inverse timescale) that is bounded from below by the quantity $\rho_c = -\sup_{\mathbf{x},t} \lambda_{\max}(\mathbf{J}_s(\mathbf{x}, t))$. This has the consequence that all trajectories converge to a single solution exponentially in time [5]. An important extension is partial contraction theory that allows to prove that the solution converge exponentially to an invariant subspace of the dynamics. (See [8] for further details).

5 Stability analysis for different navigation scenarios

In the following we list several stability results that were obtained by applying contraction theory to scenarios corresponding to system dynamics with different degrees of complexity.

1) Control of step phase without position control: The simplest case is a control of the phase on the walkers without simultaneous position control, resulting in a synchronization of the characters. The underlying dynamics is given by (4) with $m_d = 0$. Sufficient conditions for global stability of this system were derived in [8] and are given by $k > 1/\lambda^+_{\mathbf{L}}$, where $\lambda^+_{\mathbf{L}}$ is the smallest non-zero eigenvalue of symmetric part of the Laplacian $(\mathbf{L}^{\phi})_s \geq 0$. Different topologies of the coupling graphs result in different stability conditions, since for example $\lambda^+_{\mathbf{L}} = 2(1 - \cos(2\pi/N))$ for symmetric ring coupling, while $\lambda^+_{\mathbf{L}} = N$ for all-to-all coupling. (N is the number of characters.)

2) Speed control by variation of step frequency: The dynamics of this system is given by equations (2) and (3) for $m_z = 0$. The Jacobian of this system is given by $\mathbf{J} = -m_d\mathbf{L}^d\mathbf{D}_g$, where $(\mathbf{D}_g)_{ii} = g(\phi_i) > 0$ is strictly positive diagonal matrix. The conditions for global stability are: $\mathbf{J}_s = -m_d\mathbf{B} < 0$, introducing $\mathbf{B} = \mathbf{L}^d\mathbf{D}_g + \mathbf{D}_g(\mathbf{L}^d)^T$. Using diagonal stability theory [9], it can be proven that the system is contracting and globally stable for any $(\mathbf{L}^d)_s \geq 0$ and $m_d > 0$, and especially for positive coupling coefficients $K_{ij} > 0$.

In order to suppress oscillatory fluctuations of the positions of the characters the above dynamics can be extended by a low-pass filtering of the characters positions: $\dot{\mathbf{z}}(t) = -\alpha\mathbf{z}(t) + G(\boldsymbol{\phi}(t))$. By introduction of a linear coordinate transformation, bounds for the positive filter constant α can be derived that guarantee globally stable behavior of the system. The sufficient stability condition is $\frac{m_d}{\alpha}\mathbf{B} > 0$, which is true for $g(\boldsymbol{\phi}) > 0$, $m_d > 0$, $\alpha > 0$, and for $(\mathbf{L}^d)_s \geq 0$, and $\alpha^2/m_d > \frac{1}{2}\lambda_{max}(\mathbf{B})$. This results in a lower bound for the filter coefficient $\alpha > \sqrt{m_d \max g(\boldsymbol{\phi})\lambda_{max}(\mathbf{L}_s^d)}$.

An illustration of these stability bounds if given by the [**Movie_1**]; showing convergent behavior when the contraction condition $\mathbf{L}_s^d \geq 0$ is satisfied, while [**Movie_2**]. shows the divergent behavior when this condition is violated.

3) Stepsize control combined with a control of step phase: The dynamics is given by equations (3) and (4) with $m_d = 0$. It can be shown that the dynamics for $\mathbf{z}(t)$ is partially contracting for any external input $\boldsymbol{\phi}(t)$ if $\mathbf{L}^z \geq 0$. This defines a case of a hierarchical system [5], where the effective relaxation time is determined by equation (4). This subsystem is partially contracting, if $(\mathbf{L}^\phi)_s \geq 0$. In this case, the effective relaxation time for phase synchronization is given by $1/\tau_\phi^{relax} = k\lambda^+_{\mathbf{L}^\phi}$, where $\lambda^+_{\mathbf{L}^\phi}$ is the smallest non-zero eigenvalue of $(\mathbf{L}^\phi)_s$. This implies that the relaxation time for the distance control is determined by the maximum of τ_ϕ^{relax} and τ_z^{relax}, where $1/\tau_z^{relax} = m_z \min(g(\phi))\lambda^+_{\mathbf{L}^\mathbf{z}}$.

Demonstrations of this control dynamics satisfying the contraction conditions are shown in [**Movie_3**], without control of step phase, and in [**Movie_4**], with control of step phase.

4) Advanced scenarios: A simulation of a system with stable dynamics with both types of speed control (via stepsize and step frequency) and step phase control is shown in [**Movie_5**]. Using the same dynamics, a larger crowd simulated with the open-source animation engine Horde3d is shown in [**Movie_6**]. In this scenario, dynamic obstacle avoidance and control of heading direction were activated in an initial time interval for unsorting of a formation of characters. In a second time interval navigation is deactivated, and speed and position control according to the discussed principles takes over. The development of stability bounds and estimates of relaxation times for such advanced scenarios is the goal of ongoing and future extensions of the proposed approach.

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