2010 IEEE-RAS International Conference on Humanoid Robots
Nashville, TN, USA, December 6-8, 2010

Design of the dynamic stability properties of the collective behavior of articulated bipeds

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Abstract— **The control of the collective behavior of multiple interacting agents is a challenging problem in robotics and autonomous systems design. Such behaviors can be characterized by the dynamic interaction between multiple locomoting bipeds with highly nonlinear articulation dynamics. The analysis and design of the stability properties of such complex multi-component systems is a largely unsolved problem. We discuss a first approach to this problem exploiting concepts from Contraction Theory, a recent framework for the analysis of the stability of complex nonlinear dynamical systems. We demonstrate the application of this framework to groups of humanoid agents interacting collectively in different ways, requiring different types of control rules for their propagation in space and their articulation dynamics. We illustrate the framework based on a learning-based realtime-capable architecture for simulation of the kinematics of propagating bipeds, suitable for the reproduction of natural locomotion trajectories and walking styles. Exploiting central theorems from Contraction Theory and nonlinear control, we derive conditions guaranteeing the global exponential stability of the formation of the coordinated multiagent behavior. In addition, we demonstrate that the same approach permits to derive bounds that guarantee minimum convergence speeds for the formation of ordered states for collective behaviors of multiple humanoid agents.**

Index Terms— **walking bipeds, crowd steering, coordination, distributed control, self-organization, stability.**

I. Introduction

Human movements and the collective behavior of interacting characters in crowds can be described by nonlinear dynamical systems, e.g.: [1], [2]. The design of stability properties of multiagents systems is a challenging task in control theory and robotics. Especially for humanoids agents, the reason is the complexity of the dynamical systems that are required for the accurate modelling of human body movements, and even more for the interaction between multiple interacting agents.

Path planning for multi-agent systems has been studied extensively in robotics, primarily for cooperative tasks of multiple robots with relatively simple platform dynamics. The approach of the centralized planners, which is designing the motion of all agents through space-time [3], has exponential computational complexity in the number of agents. And it is not appropriate for large groups, where agents optimize for personal goals (like mutual avoidance behavior) in a presence of global common navigation goal. In contrast to this approach, decoupled planning strategies, where agents plan their behaviors individually, require priority schemes to fix conflicts between the different plans [4].

Another alternative is to use local planners for obstacle avoidance and goal-directed navigation, but to control globally the homogeneous interaction forces among agents in order to re-coordinate them by self-organization after the accomplishment of the individual subgoals [5], [6]. Such re-coordination is required to realize controllable and steerable crowds that pursue common global navigational tasks.

Some works on self-organization in systems of dynamically coupled agents have been inspired by observations in biology. These works show that coordinated behavior of large groups of agents, such as flocks of birds, can be modelled as emergent behavior arising from the dynamical coupling between interacting agents, without requiring an external central mechanism ensuring coordination [7], [8], [9], [10]. These biological observations have inspired a variety of approaches in robotics. Group coordination and cooperative control have been studied in the context of the navigation of groups of vehicles [11], and also with the objective goal to generate collective behavior by self-organization, including spontaneous adaptation to perturbations or changes in the number of agents [12]. In addition, the dynamics of interactive group behavior has been extensively studied in the field of computer animation [13], [14], [15], [16]. Specifically, some recent studies have tried to learn interaction rules from the behavior of real human crowds [17], [18], [19]. Other recent work has tried to optimize interaction behavior in crowds by exhaustive search of the parameter space exploiting computer simulations by definition of appropriate cost functions (e.g. [20]). However, most of the existing approaches for the control of group motion in computer graphics have not taken into account the effects of articulation during locomotion on the control dynamics [21], [22], [23].

Distributed control theory has started to study the temporal and spatial self-organization of crowds of agents by design of appropriate dynamic interactions, typically assuming rather simple and often even linear agent models (e.g. [24], [25], [26]). However, humanoid agents are characterized by highly complex kinematic and even dynamic properties, c.f. [27], [5]. This raises the question how approaches for the stability design of such complex dynamical systems can be developed.

This paper presents some first attempts to address this problem for simplified, but highly nonlinear dynamical

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models of human agents. We propose Contraction Theory ([28], [29]) as a central framework that makes such stability problems tractable even for agents with multiple coupled levels of control. Contraction Theory provides a useful tool specifically for modularity-based stability analysis and design [30], [31].

Opposed to classical stability analysis for nonlinear systems, Contraction Theory permits to utilize stability results for system components to derive conditions that guarantee the stability of the overall system. In this way, Contraction Theory is suitable for the derivation of conditions for the uniform exponential convergence of complex nonlinear systems.

In this paper, our new approach is demonstrated for a number of simple scenarios including interactions between multiple locomoting humanoid agents. These basic scenarios are implementing simultaneous control of distance and gait phase of the interacting bipeds.

Our work is based on a learning-based architecture that approximates complex human behavior by relatively simple nonlinear dynamical systems, which was developed previously [32], [33]. Consistent with related approaches in robotics [34], [35], [36], and related approaches motivated from biology [37], [38], [39], [40], this method generates complex movements by the combination of the learned movement primitives [41], [32]. The resulting system architecture is rather simple, making it suitable for a mathematical treatment of dynamical stability properties.

The paper is structured as follows: The structure of the animation system is sketched in section II. The dynamics underlying navigation control is described in section III. Subsequently, in section IV we introduce some basic ideas from Contraction Theory. The major results of our stability analysis and some demos of their applications to the control of crowds are described in section V, followed by the conclusions.

II. System architecture

Our investigation of the collective dynamics of crowds was based on a learning-based animation system [32] (see Fig. 1). Based on motion capture data, we learned spatio-temporal components of sets of different gait types, applying an algorithm for translation-invariant blind source separation [41]. The obtained source components were generated online by nonlinear dynamical systems, whose state dynamics was given by limit cycle oscillators. The mappings σ_j between the stable solutions of the nonlinear oscillators and the required source functions were learned by application of kernel methods. Each biped agent is modelled by a single Andronov-Hopf oscillator [32], whose solution is mapped nonlinearly onto three source signals. These signals were then superimposed with different linear weights w_{ij} and phase delays τ_{ij} in order to generate the joint angle trajectories $\xi_i(t)$ (see Fig. 1). By blending of the mixing weights and the phase delays, intermediate gait styles were generated. This allowed us to simulate specifically walking along paths with different curvatures, changes in step length

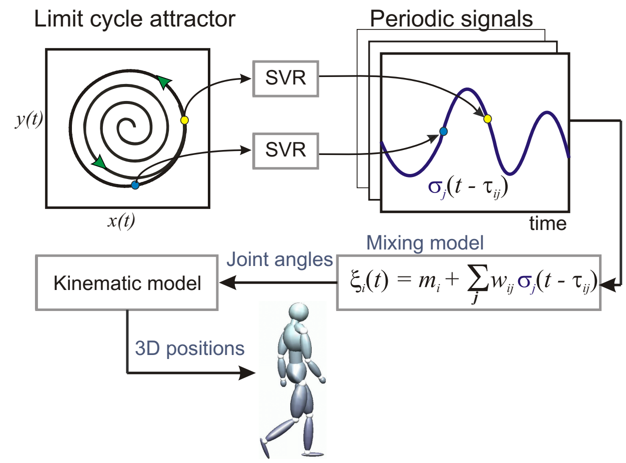

Fig. 1. Architecture of the simulation system.

and walking style. Interactive behavior of multiple agents can be modelled by making the states of the oscillators and the mixing weights dependent on the behavior of the other agents. Such couplings result in a highly nonlinear system dynamics.

For the scenarios discussed in this paper, the propagation dynamics of the bipeds was controlled, while the agents walk along parallel straight lines towards distant goal points. For the mathematical stability analysis presented in the following, we neglected the influence of the dynamics of the control of heading direction, focusing on the order formation scenarios when the agents' heading directions are already aligned. In this case, the positions of the agents can be described by a single position variable $z(t)$. (See Fig. 2). An extension of the developed analysis framework including the control of the heading direction is in progress.

III. Control dynamics

Beyond the control of heading direction, the analyzed scenarios of order formation in a group of bipeds require the control of the following variables: 1) phase within the step cycle, 2) step length, and 3) gait frequency.

The dynamics of each individual agent was modelled by an Andronov-Hopf oscillator with constant equilibrium amplitude ($r_i^* = 1$). For appropriate choice of parameters, these nonlinear oscillators have a stable limit cycle that corresponds to a circular trajectory in phase space [42].

In polar coordinates and with the instantaneous eigenfrequency ω this dynamics is given by: $\dot{r}(t) = r(t)\left(1 - r^2(t)\right)$, $\dot{\phi}(t) = \omega$. Control affects the instantaneous eigenfrequency ω of the Andronov-Hopf oscillators and their phases ϕ, while the first equation guarantees that the state stays on the limit cycle ($r(t) = 1, \forall t$).

The position z_i of each agent along the parallel paths (see Fig. 2) fulfills the differential equation $\dot{z}_i(t) = \dot{\phi}_i g(\phi_i)$, where the positive function g determines the propagation speed of the agent depending on the phase within the gait cycle. This nonlinear function was determined empirically

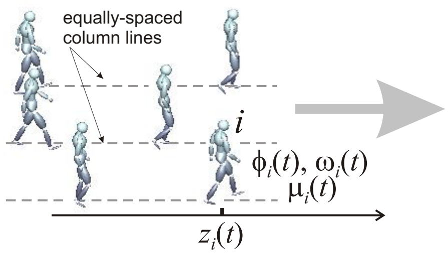

Fig. 2. Crowd coordination scenario. Every agent i is characterized by its position $z_i(t)$, the phase $\phi_i(t)$ and the instantaneous eigenfrequency $\omega_i(t) = \dot{\phi}_i(t)$ of the corresponding Andronov-Hopf oscillator, and a step-size scaling parameter $\mu_i(t)$.

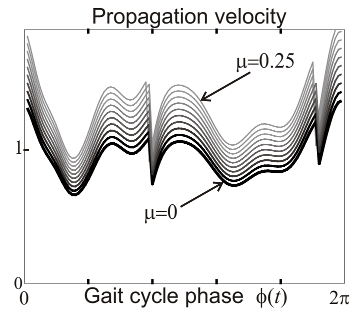

Fig. 3. Propagation velocity for 10 different values the of step length morphing parameter $\mu = [0 \ldots 0.25]$ dependent on gait cycle phase $\phi(t)$ and $\omega(t) = 1$. The vertical axis is scaled in order to make all average velocities equal to one for $\mu = 0$ (lowest thick line). This empirical estimates are well approximated by $(1 + \mu)g(\phi(t))$.

from a kinematic agent model. By integration of this propagation dynamics one obtains $z_i(t) = G(\phi_i(t) + \phi_i^0) + c_i$, with an initial phase shift ϕ_i^0 and some constant c_i depending on the initial position and phase of biped i, and with the monotonously increasing function $G(\phi_i) = \int_0^{\phi_i} g(\phi)d\phi$, assuming $G(0) = 0$. Three control rules described:

I) Control of step frequency: A simple form of speed control is based on making the frequency of the oscillators $\dot{\phi}_i$ dependent on the behavior of the other agents. Let ω_0 be the equilibrium frequency of the oscillators without interaction. Then a simple controller is defined by the differential equation

$$\dot{\phi}_i(t) = \omega_0 - m_d \sum_{j=1}^{N} K_{ij}[z_i(t) - z_j(t) - d_{ij}] \qquad (1)$$

The constants d_{ij} specify the stable pairwise relative distances in the formed order for each pair (i, j) of agents. The elements of the link adjacency matrix $\mathbf{K}$ are $K_{ij} = 1$ if agents i and j are coupled and zero otherwise. In addition, we assume $K_{ii} = 0$. The constant $m_d > 0$ defines the coupling strength.

With the Laplacian $\mathbf{L}^d$ of the coupling graph, which is defined by $L^d_{ij} = -K_{ij}$ for $i \neq j$ and $L^d_{ii} = \sum_{j=1}^{N} K_{ij}$, and the constants $c_i = -\sum_{j=1}^{N} K_{ij} d_{ij}$ the last equation system can be re-written in vector form:

$$\dot{\boldsymbol{\phi}} = \omega_0 \mathbf{1} - m_d(\mathbf{L}^d G(\boldsymbol{\phi} + \boldsymbol{\phi}^0) + \mathbf{c}) \qquad (2)$$

II) Control of step length: Step length was varied by morphing between gaits with short and long steps. A detailed analysis shows that the influence of step length on the propagation could be well captured by simple linear rescaling. If the propagation velocity of agent i is $v_i(t) = \dot{z}_i(t) = \dot{\phi}_i(t)g(\phi_i(t)) = \omega_i(t)g(\phi_i(t))$ for the normal step size, then the velocity for modified step size was well approximated by $v_i(t) = \dot{z}_i(t) = (1 + \mu_i)\omega_i(t)g(\phi_i(t))$ with the morphing parameter μ_i. The range of morphing parameters was restricted to the interval $-0.5 < \mu_i < 0.5$, where this linear scaling law was fulfilled with high accuracy. The empirically estimated propagation velocity in heading direction, dependent on gait phase, is shown in Fig.3 for different values of the step length morphing parameter μ_i.

Using the same notations as in equation (1), this motivates the definition of the following dynamics that models the influence of the step length control on the propagation speed:

$$\dot{\mathbf{z}} = \boldsymbol{\omega} g(\boldsymbol{\phi} + \boldsymbol{\phi}^0)(1 - m_z(\mathbf{L}^z \mathbf{z} + \mathbf{c})) \qquad (3)$$

In this equation $\mathbf{L}^z$ signifies the Laplacian of the relevant coupling graph, and m_z the strength of the coupling. For uncoupled agents ($m_z = 0$) this equation is consistent with the the definition of propagation speed that was given before.

III) Control of step phase: By defining separate controls for step length and step frequency it becomes possible to dissociate the control of position and step phase of the bipeds. Specifically, it is interesting to introduce a controller that results in phase synchronization between different characters. This can be achieved by addition of a simple linear coupling term to equation (1), written in vector form:

$$\dot{\boldsymbol{\phi}} = \omega_0 \mathbf{1} - m_d(\mathbf{L}^d G(\boldsymbol{\phi} + \boldsymbol{\phi}^0) + \mathbf{c}) - k\mathbf{L}^{\phi}\boldsymbol{\phi} \qquad (4)$$

with $k > 0$ and the Laplacian $\mathbf{L}^{\phi}$. (All sums or differences of angular variables were computed by modulo 2π).

The mathematical results derived in the following section apply to subsystems of the complete system dynamics that is given by equations (3) and (4). In addition, simulations are presented for the full system dynamics.

IV. Contraction Theory

Dynamical systems describing the behavior of autonomous agents are essentially nonlinear. In contrast to the linear dynamical systems, a major difficulty of the analysis of stability properties of nonlinear is that the stability properties of parts usually do not transfer to composite systems. Contraction Theory [28] provides a general method for the analysis of essentially nonlinear systems, which permits such a transfer, making it suitable for the analysis of complex systems with many components. Contraction Theory characterizes the system stability by the behavior of the differences between solutions with different initial conditions. If these differences vanish exponentially over time, all solutions converge towards a single trajectory, independent from the initial states.

In this case, the system is called *globally asymptotically stable*. For a general dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \quad (5)$$

assume that $\mathbf{x}(t)$ is one solution of the system, and $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) + \delta\mathbf{x}(t)$ a neighboring one with a different initial condition. The function $\delta\mathbf{x}(t)$ is also called *virtual displacement*. With the Jacobian of the system $\mathbf{J}(\mathbf{x}, t) = \frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial \mathbf{x}}$ it can be shown [28] that any nonzero virtual displacement decays exponentially to zero over time if the symmetric part of the Jacobian $\mathbf{J}_s = (\mathbf{J} + \mathbf{J}^T)/2$ is uniformly negative definite, denoted as $\mathbf{J}_s < 0$. This implies that it has only negative eigenvalues for all relevant state vectors $\mathbf{x}$. In this case, it can be shown that the norm of the virtual displacement decays at least exponentially to zero, for $t \to \infty$. If the virtual displacement is small enough, then

$$\dot{\delta\mathbf{x}}(t) = \mathbf{J}(\mathbf{x}, t)\delta\mathbf{x}(t)$$

implying through $\frac{d}{dt}||\delta\mathbf{x}(t)||^2 = 2\delta\mathbf{x}^T(t)\mathbf{J}_s(\mathbf{x}, t)\delta\mathbf{x}$ the inequality: $||\delta\mathbf{x}(t)|| \leq ||\delta\mathbf{x}(0)||\, e^{\int_0^t \lambda_{\max}(\mathbf{J}_s(\mathbf{x},s))\, ds}$. This implies that the virtual displacements decay with a *convergence rate* (inverse timescale) that is bounded from below by the quantity $\rho_c = -\sup_{\mathbf{x},t} \lambda_{\max}(\mathbf{J}_s(\mathbf{x}, t))$, where $\lambda_{\max}(.)$ signifies the largest eigenvalue. With $\rho_c > 0$ all trajectories converge to a single solution exponentially in time [28].

Contraction analysis can be applied also to hierarchically coupled systems [28]. Consider a composite dynamical system with two components, where the dynamics of the first subsystem is not influenced by the dynamics of the second one. Such system is called *hierarchically coupled*. If the first subsystem does not depend on the second, its state exponentially converges to attractor solution if the symmetrized Jacobian of the first subsystem is negative definite. Then, in case of bounded interaction, the first subsystem introduces an exponentially decaying disturbance for the second subsystem. In this case, (see [28] for details of proof), the uniformly negative definite symmetrized Jacobian of the second subsystem implies exponential convergence of its state to an exponentially decaying ball in phase space. The whole system is then globally exponentially convergent to a single trajectory.

Many systems are not contracting with respect to all dimensions of the state space, but show convergence with respect to a subset of dimensions. Such behavior can be mathematically characterized by *partial contraction* [31], [33]. The underlying idea is to construct an auxiliary system that is contracting with respect to a subset of dimensions (or submanifold) in state space. The major result is the following:

Theorem 1: Consider a nonlinear system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t) \quad (6)$$

and assume that the auxiliary system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, t) \quad (7)$$

is contracting with respect to $\mathbf{y}$ uniformly for all relevant $\mathbf{x}$. If a particular solution of the auxiliary system verifies

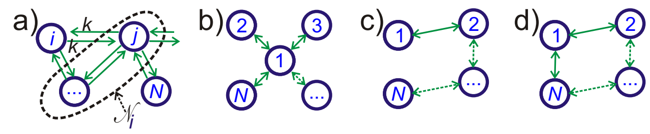

Fig. 4. a. Symmetric coupling with coupling constant k, $\mathcal{N}_i$ specifying the set of neighbors of i; b.-d. star, chain, ring coupling schemes.

a specific smooth property, then trajectories of the original system (6) verify this property exponentially. The original system is then said to be partially contracting. [31].

A 'smooth property' is a property of the solution that depends smoothly on space and time, such as convergence against a particular solution or a properly defined distance to a subspace in phase space. The proof of the theorem is immediate noticing that the observer-like system (7) has $\mathbf{y}(t) = \mathbf{x}(t)$ for all $t \geq 0$ as a particular solution. Since all trajectories of the $\mathbf{y}$-system converge exponentially to a single trajectory, this implies that also the trajectory $\mathbf{x}(t)$ verifies this specific property with exponential convergence.

It is thus sufficient to show that the auxiliary system is contracting in order to prove the convergence to a subspace. Let us assume that system has a flow-invariant *linear subspace* $\mathcal{M}$, which is defined by the property that trajectories starting in this space always remain in it for arbitrary times ($\forall t$: $\mathbf{f}(\mathcal{M}, t) \subset \mathcal{M}$). If matrix $\mathbf{V}$ is an orthonormal projection onto $\mathcal{M}^{\perp}$, then sufficient condition for global exponential convergence to $\mathcal{M}$ is: $\mathbf{V}\mathbf{J}_s\mathbf{V}^T < \mathbf{0}$, where smaller sign indicates that this matrix is negative definite (see [29], [33]).

V. Exemplary results

We derived contraction bounds for three scenarios that correspond to control dynamics with increasing levels of complexity.

1) Control of step phase without position control: The simplest case is a control of the phase within the step cycle of the walkers without simultaneous control of the position of the agents. Such simple control already permits to simulate interesting behaviors, such as soldiers synchronizing their step phases [33], [**Demo**[1]]. The underlying dynamics is given by (4) with $m_d = 0$. For N identical dynamical systems, with symmetric identical coupling gains k this dynamics can be written

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + k \sum_{j \in \mathcal{N}_i} (\mathbf{x}_j - \mathbf{x}_i), \quad \forall i = 1, \ldots, N \quad (8)$$

where $\mathcal{N}_i$ defines the index set specifying the neighborhood in the coupling graph, i.e. the other subsystems or agents that are coupled with agent i (see Fig.4 for examples).

This type of symmetric coupling, where the interaction forces between subsystems depend only on the differences of the phase variables is called *diffusive coupling*. In this case, the *Laplacian matrix* of the coupling scheme is given by $\mathbf{L} = \mathbf{L}_G \bigotimes I_p$, where p is the dimensionality of the

[1] www.uni-tuebingen.de/uni/knv/arl/avi/huma/video0.avi

individual sub-systems, and where $\bigotimes$ signifies the *Kronecker product*. The Laplacian of the coupling graph is the matrix $\mathbf{L}_G$. The system then can be rewritten compactly as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) - k\mathbf{L}\mathbf{x}$ with the concatenated phase variable $\mathbf{x} = [\mathbf{x}_1^T, ..., \mathbf{x}_n^T]^T$. The Jacobian of this system is $\mathbf{J}(\mathbf{x}, t) = \mathbf{D}(\mathbf{x}, t) - k\mathbf{L}$, where the block-diagonal matrix $\mathbf{D}(\mathbf{x}, t)$ has the Jacobians of the uncoupled components $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_i, t)$ as entries.

The dynamics has a flow-invariant linear subspace $\mathcal{M}$ that contains the particular solution $\mathbf{x}_1^* = \cdots = \mathbf{x}_n^*$. For this solution all state variables $\mathbf{x}_i$ are identical and thus in synchrony. In addition, for this solution the coupling term in equation (8) vanishes, so that the form of the solution is identical with the solution of the uncoupled systems $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i)$. If $\mathbf{V}$ is a projection matrix onto the subspace $\mathcal{M}^{\perp}$, then, the sufficient condition for convergence toward $\mathcal{M}$ is $\mathbf{V}(\mathbf{D}(\mathbf{x}, t) - k\mathbf{L})_s\mathbf{V}^T < \mathbf{0}$, [29], [33], where smaller sign indicates that the matrix is negative definite. This implies

$$\lambda_{\min}\left(\mathbf{V}(k\mathbf{L})_s\mathbf{V}^T\right) = k\lambda_{\mathbf{L}}^+ > \sup_{\mathbf{x},t} \lambda_{\max}\left(\mathbf{D}_s\right)$$

with $\lambda_{\mathbf{L}}^+$ being the smallest non-zero eigenvalue of symmetrical part of the Laplacian $\mathbf{L}_s$. The maximal eigenvalue for the individual oscillator is $\sup_{\mathbf{x},t} \lambda_{\max}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, t)\right)$. The sufficient condition for global stability of the overall system is given by $k > \sup_{\mathbf{x},t} \lambda_{\max}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, t)\right)/\lambda_{\mathbf{L}}^+$. This implies the following minimum convergence rate: $\rho_c = -\sup_{\mathbf{x},t} \lambda_{\max}(\mathbf{V}(\mathbf{D}(\mathbf{x}, t) - \mathbf{L})_s\mathbf{V}^T)$.

For the special case of (4) with $m_d = 0$ this implies the sufficient contraction conditions $k > 0$ and $(\mathbf{L}^{\phi})_s \geq 0$.

Different topologies of the coupling graphs result in different stability conditions, since for example $\lambda_{\mathbf{L}}^+ = 2(1 - \cos(2\pi/N))$ for symmetric ring coupling, and $\lambda_{\mathbf{L}}^+ = N$ for all-to-all coupling. (N is the number of agents.) See [31] and [33] for details.

In order to validate these theoretical bounds we computed empirical convergence rates ρ^{exper} from our simulated system. The rates were obtained by analyzing the time courses of the virtual displacements $||\delta x|| \sim e^{-\rho^{\text{exper}}t}$ and approximating them by exponential convergence. The norm of the virtual displacement $||\delta x||$ was approximated by the angular dispersion $\hat{R} = (1 - \frac{1}{N}|\sum_j e^{i\phi_j}|)^{\frac{1}{2}}$ of the phases ϕ_j of Andronov-Hopf oscillators (c.f. [43]), averaged over many simulations with random initial conditions. After an initial time interval (offset time), the dispersion shows exponential decay with time. Convergence rates of this decay were estimated by linear regression from the logarithm of dispersion against time, averaged over many trials.

The comparison results for different coupling topologies and for different numbers of agents are shown in Fig.5 for the case of linear diffusive couplings of Andronov-Hopf oscillators in Euclidian space [33]. Figure 5a) shows the dependence between the coupling strengths k and the convergence rate ρ^{exper} of the angular dispersion in the scenario of phase synchronization, estimated from simulations in the regime of the exponential convergence. As derived from the

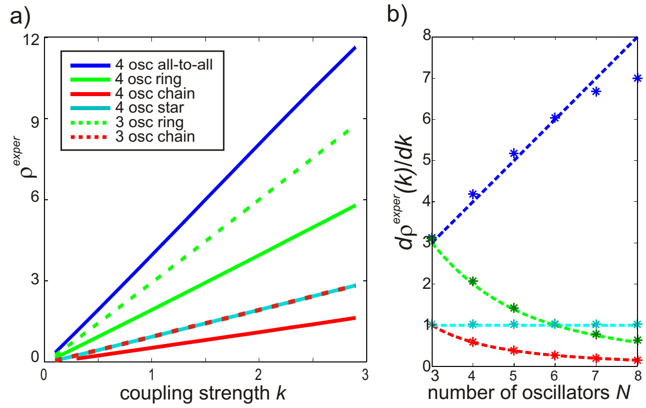

Fig. 5. a. The relationship between convergence rate and coupling strength k for different types of coupling graphs; b. Slopes of this relationship as function of the number N of Hopf oscillators, comparing simulation results (indicated by asterisk symbols near to the lines) and derived from the theoretical bounds [33].

theoretical bound, the convergence rate varies linearly with the coupling strength. In the case of three oscillators the ring coupling is equivalent with all-to-all coupling. Figure 5b) shows the slope $d\rho^{\text{exper}}(k)/dk$ of this linear relationship as function of N, the number of oscillators in the network. We find a close similarity between the theoretically predicted relationship (dashed curves) and the results from the simulation (indicated by the stars). In addition, it is evident that for all-to-all coupling the convergence rate increases with the number of oscillators, while for chain or ring coupling the convergence speed decreases with the number of oscillators (for fixed coupling strength). These results show in particular that the proposed theoretical framework is not only suitable for proving asymptotic stability, but also for guaranteeing the convergence speed of the system dynamics.

2) Speed control by variation of step frequency: The dynamics of this system is given by equations (2) and (3) for $m_z = 0$. Assuming arbitrary initial distances and phase offsets for different propagating agents, implying $G(\phi_i^0) = c_i$, $c_i \neq c_j$, for $i \neq j$, we redefine d_{ij} as $d_{ij} - (c_i - c_j)$ in (1), and accordingly redefine $\mathbf{c}$ in (2). Assuming this control dynamics, and two agents i and j that follow a leading agent, their phase trajectories converge to a single unique trajectory only if $c_i = c_j$. This is a consequence of the one-to-one correspondence between gait phase and position of the agent that is given by equation (2). In all other cases the trajectories of the followers converge to one-dimensional, but distinct, attractors in phase-position space that are uniquely defined by c_i. These attractors correspond to a behavior where the follower's position oscillates around the position of the leader. Below we provide the sufficient conditions for the global stability of such attractor states.

For the analysis of contraction properties we regard an auxiliary system obtained from (2) by keeping the terms which are only dependent on $\boldsymbol{\phi}$: $\dot{\boldsymbol{\phi}} = -m_d\mathbf{L}^dG(\boldsymbol{\phi} + \boldsymbol{\phi}^0)$. The symmetrized Jacobian of this system projected to the orthogonal compliment of flow-invariant linear subspace

$\phi_1^* + \phi_1^0 = \ldots = \phi_N^* + \phi_N^0$ determines whether this system is partially contracting. By virtue of a linear change of variables the study of the contraction properties of this system is equivalent to study the contraction properties of the dynamical system $\dot{\boldsymbol{\phi}} = -m_d \mathbf{L}^d G(\boldsymbol{\phi})$ on trajectories converging towards its flow-invariant manifold, the linear subspace of $\phi_1^* = \ldots = \phi_N^*$.

In order to derive an asymptotic stability condition, we consider the following auxiliary system, corresponding to a part of (2): $\dot{\boldsymbol{\phi}} = -m_d \mathbf{L}^d G(\boldsymbol{\phi})$. The Jacobian of this system is given by $\mathbf{J} = -m_d \mathbf{L}^d \mathbf{D}_g$, where $(\mathbf{D}_g)_{ii} = g(\phi_i) > 0$ is a strictly positive diagonal matrix. Exploiting diagonal stability theory [44], we introduce the Lyapunov function: $W(\boldsymbol{\phi}) = \sum_i \tilde{\mathbf{D}}_{ii} \int_0^{\phi_i} G(\tau) d\tau > 0$, where $\tilde{\mathbf{D}} > 0$ is some positive diagonal matrix and where our function $G(\phi)$ with $G'(\phi) > 0$ satisfies: $G(\phi)\phi > 0$, $G(0) = 0$. The time derivative of this Lyapunov function on the trajectories of our auxiliary dynamical system is: $\dot{W}(\boldsymbol{\phi}) = -m_d G(\boldsymbol{\phi})^T (\mathbf{L}^d \tilde{\mathbf{D}} + \tilde{\mathbf{D}} (\mathbf{L}^d)^T) G(\boldsymbol{\phi})$. The condition $\dot{W}(\boldsymbol{\phi}) \leq 0$ is fulfilled if one can determine a full rank matrix $\tilde{\mathbf{D}} > 0$, such that $(\mathbf{L}^d \tilde{\mathbf{D}} + \tilde{\mathbf{D}} (\mathbf{L}^d)^T) \geq 0$. If $\mathbf{L}^d = (\mathbf{L}^d)^T \geq 0$ one can choose $\tilde{\mathbf{D}} = \mathbf{I}$. This guarantees $\dot{W}(\boldsymbol{\phi}) = -m_d G(\boldsymbol{\phi})^T (\mathbf{L}^d + (\mathbf{L}^d)^T) G(\boldsymbol{\phi}) \leq 0$ for $m_d > 0$ and $\dot{W}(\boldsymbol{\phi}) = 0$ only if $G(\boldsymbol{\phi})$ is in the nullspace of $\mathbf{L}^d$, where $\phi_1^* = \ldots = \phi_N^*$. In this case, the auxiliary system is globally asymptotically stable and its state converges to an attractor with $\phi_1^* = \ldots = \phi_N^*$ for any initial condition assuming $(\mathbf{L}^d)_s \geq 0$ and $m_d > 0$. Thus the sufficient conditions for asymptotic stability are satisfied for all types of symmetric diffusive couplings with positive coupling strength. For the case of asymmetric coupling graphs with more general structure including negative feedback links some results on asymptotic stability have been provided in [45].

The sufficient conditions for (exponential) partial contraction towards flow-invariant subspace are: $\mathbf{V}\mathbf{J}_s(\boldsymbol{\phi})\mathbf{V}^T = -m_d \mathbf{V}\mathbf{B}(\boldsymbol{\phi})\mathbf{V}^T < 0$, introducing $\mathbf{B}(\boldsymbol{\phi}) = \mathbf{L}^d \mathbf{D}_g + \mathbf{D}_g (\mathbf{L}^d)^T$ and $\mathbf{V}$ signifying the projection matrix onto the orthogonal complement of the flow-invariant linear subspace. For diffusive coupling with symmetric Laplacian the linear flow-invariant manifold $\phi_1^* = \ldots = \phi_N^*$ is also the null-space of the Laplacian. In this case, the eigenvectors of the Laplacian that correspond to positive eigenvalues can be used to construct the projection matrix $\mathbf{V}$. For $m_d > 0$ the contraction conditions are thus satisfied if $\mathbf{V}\mathbf{B}(\boldsymbol{\phi})\mathbf{V}^T = \mathbf{V}(\mathbf{L}^d \mathbf{D}_g + \mathbf{D}_g (\mathbf{L}^d)^T)\mathbf{V}^T > 0$ for any diagonal matrix $\mathbf{D}_g > 0$.

Next we prove the exponential contraction conditions for the particular case of symmetrical all-to-all coupling. In this case $\mathbf{L}^d = N\mathbf{I} - \mathbf{1}\mathbf{1}^T \geq 0$, where $\mathbf{I}$ is identity matrix of size N. Since $\mathbf{V}\mathbf{1} = \mathbf{1}^T \mathbf{V}^T = 0$, we obtain $\frac{1}{2}\mathbf{V}(\mathbf{L}^d \mathbf{D}_g + \mathbf{D}_g (\mathbf{L}^d)^T)\mathbf{V}^T = N\mathbf{V}(\mathbf{D}_g)\mathbf{V}^T > 0$ for $\mathbf{D}_g > 0$. A lower bound for the contraction rate is computed from the projected symmetrized Jacobian $\mathbf{V}\mathbf{J}_s(\boldsymbol{\phi})\mathbf{V}^T = -\frac{m_d}{2}\mathbf{V}\mathbf{B}(\boldsymbol{\phi})\mathbf{V}^T$. This results in the guaranteed contraction rate $\rho_{min} = m_d \min_\phi (g(\phi)) \lambda^+_{\mathbf{L}^d}$, where for all-to-all symmetric coupling $\lambda^+_{\mathbf{L}^d} = N$.

Next we regard two additional important cases:

a) In order to reduce oscillatory fluctuations of the positions of the follower around the leader (see above), the dynamics of the former can be extended by a **low pass filtering**: $\dot{\mathbf{z}}(t) = -\alpha \mathbf{z}(t) + G(\boldsymbol{\phi}(t))$. By introduction of a linear coordinate transformation, bounds for the positive filter constant α can be derived that guarantee globally stable behavior of the system. The dynamics of the system with low-pass filtering is given by

$$\begin{cases} \dot{\boldsymbol{\phi}} = -\, m_d \mathbf{L}^d \mathbf{z} \\ \dot{\mathbf{z}} = -\, \alpha \mathbf{z} + G(\boldsymbol{\phi}) + \mathbf{c} \end{cases}$$

This system can be further transformed linearly by introduction of the new coordinates $\mathbf{y} = -(m_d/\alpha)\mathbf{L}^d \mathbf{z}$ and $\mathbf{x} = \mathbf{y} + \boldsymbol{\phi}$. In these new coordinates the Jacobian of the system is $\mathbf{J}(\boldsymbol{\phi}) = -m_d/\alpha \begin{bmatrix} \mathbf{L}^d \mathbf{D}_g & -\mathbf{L}^d \mathbf{D}_g \\ \mathbf{L}^d \mathbf{D}_g & (\alpha^2/m_d)\mathbf{I} - \mathbf{L}^d \mathbf{D}_g \end{bmatrix}$. The symmetrized Jacobian does not contain the off-diagonal block terms. This implies that two conditions are sufficient to guarantee that the system is partially contracting: The first condition, which is identical to the case without filtering, is given by $(m_d/\alpha)\mathbf{V}\mathbf{B}\mathbf{V}^T > 0$. This is fulfilled if $g(\boldsymbol{\phi}) > 0$, $m_d > 0$, $\alpha > 0$, and $(\mathbf{L}^d)_s \geq 0$ for all-to-all symmetric couplings. The second condition is $\alpha^2/m_d > \frac{1}{2}\lambda_{max}(\mathbf{V}\mathbf{B}\mathbf{V}^T)$ with $(\mathbf{L}^d)_s \geq 0$. This inequality results in a sufficient lower bound for the filter coefficient $\alpha > \sqrt{m_d \max_\phi (g(\phi)) \lambda_{max}((\mathbf{L}^d)_s)}$.

An illustration of these stability bounds if given by the [**Demo**[2]]; that shows convergent behavior of the bipeds when the contraction condition $m_d > 0, (\mathbf{L}^d)_s \geq 0$ is satisfied for all-to-all coupling. [**Demo**[3]] shows the divergent behavior of a group when this condition is violated when $m_d < 0$.

b) The same proof can be extended for **nonlinear control rules**. In this case the eigenfrequency is given by a nonlinear modification of the control rule in eq. (1), for agent i coupled to agent j as: $\omega_i = \omega_0 + m_d h(z_j - z_i + d_{ij})$, where the saturating nonlinear function h could be given, for example by $h(z) = 1/[1 + \exp(-\gamma z)]$ with $\gamma > 0$. This nonlinear function limits the range of admissible speeds for the controller. Using the same notations as above, the dynamics of a single follower that follows a leader at position $P(t)$ is given by: $\dot{\phi}(t) = \omega_0 + m_d h(P(t) - G(\phi(t)) + c)$. The Jacobian of this dynamics $\mathbf{J}_s = -m_d h' g(\phi) < 0$ is negative, which follows from $m_d > 0$, $g(\phi) > 0$ and taking into account $h'(z) = dh(z)/dz > 0, \forall z$, what guarantees contraction.

Again this dynamics can be extended for N agents, resulting in the nonlinear differential equation system: $\dot{\phi}_i(t) = \omega_0 - m_d \sum_{j=1}^N K_{ij} h(G(\phi_i) - G(\phi_j) + d_{ij}), \forall i$. The Jacobian of the system is: $\mathbf{J}(\boldsymbol{\phi}) = -m_d \mathbf{L}^d(\boldsymbol{\phi})\mathbf{D}_g$, where $\mathbf{L}^d_{ij}(\boldsymbol{\phi}) = -K_{ij} h'(G(\phi_i) - G(\phi_j) + d_{ij})$, $\mathbf{L}^d_{ii}(\boldsymbol{\phi}) = \sum_{j \neq i}^N K_{ij} h'(G(\phi_i) - G(\phi_j) + d_{ij})$, $d_{ii} = 0$, ($\mathbf{D}_g$ is defined as before). Furthermore, the even function $h'(z) > 0$ implies that the Laplacian $\mathbf{L}^d(\boldsymbol{\phi})$ is symmetric diagonally dominant and it stays positive semidefinite for

[2] www.uni-tuebingen.de/uni/knv/arl/avi/huma/video1.avi
[3] www.uni-tuebingen.de/uni/knv/arl/avi/huma/video2.avi

any positive $K_{ij} > 0$, by Gershgorin's Theorem [46]. This implies that the system is asymptotically stable, its solutions converging to an attractor. The analysis of exponential convergence requires further steps that exceed the scope of this paper.

3) Stepsize control combined with a control of step phase: The dynamics is given by equations (3) and (4) with $m_d = 0$. This dynamics defines a hierarchically coupled system (Section IV, [28]). The dynamics for $\mathbf{z}(t)$ given by equation (3) is partially contracting in case of all-to-all coupling for any bounded external input $\boldsymbol{\phi}(t)$, if $m_z > 0$, $\mathbf{L}^z \geq 0$ and $\boldsymbol{\omega}(t) > 0$. These sufficient contraction conditions can be derived from the requirement of the positive-definiteness of the symmetrized Jacobian applying a similar technique as above. The Jacobian of this subsystem is $\mathbf{J}(\boldsymbol{\phi}, \boldsymbol{\omega}) = -m_z \mathbf{D}_g^z(\boldsymbol{\phi}, \boldsymbol{\omega})\mathbf{L}^z$, with the diagonal matrix $(\mathbf{D}_g^z(\boldsymbol{\phi}, \boldsymbol{\omega}))_{ii} = \omega_i g(\phi_i + \phi_i^0) > 0$ that is positive definite since $g(\boldsymbol{\phi}) > 0$ and $\boldsymbol{\omega} > 0$. This subsystem is (exponentially) contracting and its relaxation rate is determined by $\rho_z = m_z \min_\phi (g(\phi))\lambda^+_{\mathbf{L}^z}$ (in the case of all-to-all coupling) for any input from the dynamics of $\phi(t)$ eq. (4). The last dynamics is contracting when $(\mathbf{L}^\phi)_s \geq 0$ and its relaxation rate is $\rho_\phi = k\lambda^+_{\mathbf{L}^\phi}$, where $\lambda^+_{\mathbf{L}^\phi}$ is the smallest non-zero eigenvalue of $(\mathbf{L}^\phi)_s$. The effective relaxation time of the overall dynamics is thus determined by the minimum of the contraction rates ρ_ϕ and ρ_z.

Demonstrations of this control dynamics satisfying the contraction conditions are shown in [**Demo**[4]], without control of step phase, and in [**Demo**[5]], with control of step phase.

4) Advanced scenarios: A simulation of a system with stable dynamics with both types of speed control (via step size and step frequency) and step phase control is shown in [**Demo**[6]]. Using the same dynamics, a larger crowd simulated with the open-source animation engine Horde3d [47] is shown in [**Demo**[7]]. In this scenario, dynamic obstacle avoidance and control of heading direction were activated in an initial time interval for unsorting of a formation of agents. In a second time interval navigation is deactivated, and speed and position control according to the discussed principles takes over. [**Demo**[8]] demonstrates a large synchronizing crowd with 36 avatars. The development of stability bounds and estimates of relaxation times for even more advanced scenarios including multiple control levels of navigation is the goal of ongoing work.

VI. Conclusions

For the example of a learning-based system for the simulation of locomoting groups, we have demonstrated first examples of an application of Contraction Theory for the analysis and the design of stability and convergence properties of collective behaviors in animated crowds. The approximation of the essential dynamic properties of the humanoid agents is critical to make the systems accessible for the analysis of stability. Obviously, future work has to extend this work to more complex scenarios and platform dynamics. A generalization of this approach to other problems in robotics, such as the control of goal-directed behavior and an extension for non-periodic movements seems possible [48], [49], [50]. Such extensions form the core of the planned future work.

[4]www.uni-tuebingen.de/uni/knv/arl/avi/huma/video3.avi
[5]www.uni-tuebingen.de/uni/knv/arl/avi/huma/video4.avi
[6]www.uni-tuebingen.de/uni/knv/arl/avi/huma/video5.avi
[7]www.uni-tuebingen.de/uni/knv/arl/avi/huma/video6.avi
[8]www.uni-tuebingen.de/uni/knv/arl/avi/huma/video7.avi

Acknowledgments

Supported by DFG Forschergruppe 'Perceptual Graphics', the EC FP7 projects 'AMARSI' and 'TANGO' and the Hermann und Lilly-Schilling-Stiftung. Authors thank Karsten Rohweder for help with animations in Horde3d [47].

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